On the Connection and Linearization Problem for Discrete Hypergeometric $q$-Polynomials

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In the present paper, starting from the second-order difference hypergeometric equation on the non-uniform lattice $x(s)$ satisfied by the set of discrete hypergeometric orthogonal $q$-polynomials $\{p_n\}$, we find analytical expressions of the expansion coefficients of any $q$-polynomial $r_m(x(s))$ on $x(s)$ and of the product $r_m(x(s))q_s(x(s))$ in series of the set $\{p_n\}$. These coefficients are given in terms of the polynomial coefficients of the second-order difference equations satisfied by the involved discrete hypergeometric $q$-polynomials.

Key Words: $q$-polynomials; connection and linearization problems.

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1. INTRODUCTION

The expansion of any arbitrary discrete polynomial $q_n(x)$ in series of a general (albeit fixed) set of discrete hypergeometric polynomial $(p_n(x))$ is a matter of great interest, solved only for some particular classical cases (for a review see [8, 9, 17] up to the middle of the seventies and [5, 40], since then up to now). This is particularly true for the deeper problem of linearization of a product of any two polynomials. Usually, the determination of the expansion coefficients in these particular cases required a deep knowledge of special functions and, at times, ingenious induction arguments based in the three-term recurrence relation of the involved orthogonal polynomials [9, 14, 16, 17, 22, 25, 34, 42–44]. Only recently have general and widely applicable strategies begun to appear [5–7, 13, 19, 24, 26, 27, 29–31, 39, 41].

One of the reasons for this increasing interest is the applications of such kind of problems in several branches of mathematics and physics. For example, Gasper in his paper [17] motivated the connection and linearization problem in the framework of the positivity. Nine years after, one of the most famous conjectures, the Bieberbach conjecture ($|a_n| \leq n$) for analytic and univalent functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $|z| < 1$, was solved by Louis de Branges using an inequality proved by Askey and Gasper in 1976 [10] in the framework of the positivity. In fact they proved that

$$\sum_{k=0}^{\infty} P_k^{\alpha,0}(t) = \frac{(\alpha + 2)^n}{n!} F_2\left(\begin{array}{c}-n, n + \alpha + 2, \frac{a+2}{2} \\ \frac{a+3}{2}, \alpha + 1 \end{array}\bigg| t\right) \geq 0, \quad 0 \leq t < 1, \alpha > -2, \quad (1.1)$$

where $(a)_n$ is the Pochhammer symbol and $P_n^{\alpha,\beta}(x)$ denotes the classical Jacobi polynomials.

Here in this work we will study the problem of finding the connection $c_{mn}$ and linearization $c_{jn,m}$ coefficients, i.e., the coefficients on the expansions [9]

$$q_m(x) = \sum_{n=0}^{m} c_{mn} p_n(x), \quad (1.2)$$

$$q_m(x) r_j(x) = \sum_{n=0}^{m+j} c_{jn,m} p_n(x), \quad (1.3)$$

respectively, where $q_m(x)$ and $r_j(x)$ are any $m$th-degree and $j$th-degree polynomials, and $(p_n)$ denotes an arbitrary set of polynomials.
Notice that since the involved hypergeometric series in (1.1) is terminating, i.e., has a finite number of terms, the above problem can be considered as a connection problem between two families of polynomials where all the connection coefficients are positive (and equal to 1 in this example). So Gasper’s words about the importance in applications of the connection and linearization problems, and the positivity of the corresponding coefficients, become very actual and of interest.

The first who considered the linearization problem for discrete polynomials (notice that in de Branges’s proof the “continuous” Jacobi polynomials have been used) was Eagleson in 1969 for Kravchuk polynomials [14]. Later, Gasper [17] studied the connection problem for the Hahn $h^{a,\beta}(x, N)$ polynomials

$$h_j^{a,\beta}(x, M) = \sum_{n=0}^{j} c_{jn} h_n^{a,\beta}(x, N), \quad j \leq \min\{N - 1, M - 1\},$$

and completely solved it (the particular case $N = M$, of interest because $c_{jn} \geq 0$, he solved one year earlier in [16]), from where, by the limiting process it is possible to obtain the connection coefficients for Jacobi polynomials as well as for other continuous and discrete families (see [16, 17] for further information on this). Some years later, Askey and Gasper [11] considered the linearization problem when the involved polynomials were the discrete polynomials of Hahn, Meixner Kravchuk, and Charlier (for a review on discrete polynomials see [32]) but only in the special case when all $r_m$, $q_j$, and $p_n$ belong to the same family with the same parameters (in [17] some preliminary results regarding the positivity of such coefficients were discussed).

In all these cases, continuous and discrete, the proofs were based on very specific characteristics of the involved families, particularly their hypergeometric representation and generating functions which have been exploited for finding the corresponding solution.

It is important to remark that, even in the case when it is possible to compute explicitly the connection or the linearization coefficients, it is not always easy to show that they are nonnegative which is important as we already pointed out. This leads to a recurrent method, i.e., to find a difference equation for the coefficients $c_{mn}$ and $c_{jmn}$, respectively, and from it deduce their nonnegativity. The first who did this was Hylleraas [22] in 1962 for a product of two Jacobi polynomials. In fact Hylleraas was able to solve the obtained recurrence relation for some special cases and prove the nonnegativity of the coefficients in some of these cases. Later, this method was used by Askey and Gasper (see, e.g., [11]) to prove the nonnegativity of the linearization coefficients for certain families of orthogonal polynomials.
More recently, Ronveaux, Zarzo, Area, and Godoy [6, 19, 39] developed a recurrent method, called the NAVIMA algorithm, for solving the connection problem (1.2) for all families of classical polynomials, as well as some special kind of linearization problem and used it for solving different problems related with the associated, Sobolev-type polynomials, etc. [20, 21]. Although they used it only for solving a very special linearization problem, it can be easily extended for solving the general problem (1.3) [13]. Let us point out that there is a very similar algorithm for finding the recurrence relation for both the connection and linearization coefficients due to Lewanowicz [26, 27, 29]. The most important tool in both the aforementioned algorithms was the structure relations (or the Al-Salam and Chihara characterization) that the polynomials \( p_n \) in (1.2) and (1.3) satisfy.

Both problems, connection and linearization, are of great interest also in physics. For example, for the \( 2^l \)-pole transitions in hydrogen-like atoms (and other related systems) the radial part of the probability is proportional to integrals of the form

\[
T^{12}_l = \int_0^{\infty} \left[ L^{2l+1}_{n_1} (\alpha_1 r) L^{2l+1}_{n_2} (\alpha_2 r) \right] r^m e^{-\tau} \, dr,
\]

where \( L_n \) are the Laguerre polynomials. This kind of integral also appears in the theory of Morse oscillators as well as in transitions for spherical-symmetric systems [34]. Furthermore, for the spherical-symmetric case the Wigner–Ekkart theorem [15] allows us to write the matrix elements of certain irreducible operators in terms of products of two (or more) \( 3j \) symbols (Hahn and dual Hahn polynomials [32]), \( 6j \) symbols (Racah polynomials [32]), etc., as well as their \( q \)-analogues.

To conclude this introduction we need to say that in the world of \( q \)-polynomials (discrete case) there are not so many results concerning these problems. One of the first who was interested in this was Rogers [37, 38] who used a \( q \)-analogue of the connection formula for Jacobi polynomials \( P_n^{\alpha, \gamma} (x) = \sum_{l=0}^{n} \binom{n+\alpha}{l} \binom{n+\gamma}{l} (c_n)_l P_n (x) \), \( c_n \geq 0 \), for the \( q \)-ultraspherical polynomials to prove some Rogers–Ramanujan identities (see also [36]). Also, very recently, this problem was considered in [3, 4, 28] for \( q \)-polynomials in the exponential lattice \( x(s) = q^s \) [2, 32, 33], where the authors obtained recurrence relations for the coefficients in (1.2) and (1.3). Again, in these works the use of the structure relations played a fundamental role. But such relations do not exist for any arbitrary family of \( q \)-polynomials. In [2] it was proven that all families of \( q \)-polynomials on the exponential lattice \( x(s) = c_1 q^s + c_3 \) satisfy such a relation, but for the general lattice \( x(s) = c_1 q^s + c_2 q^{-s} + c_3 \) [12, 32] the problem is still open. Then the following question naturally arises: What does one do in the case when one does not have structure relations? This question was solved for the continuous case in [7, 41] and for the discrete case in [5].
The main goal of the present paper is to give an alternative method for finding the connection and linearization problem for $q$-hypergeometric polynomials obtaining explicit expressions for the coefficients $c_{mn}$ and $c_{jmn}$ in (1.2)–(1.3) in terms of the coefficients of the second order difference equation of hypergeometric-type on the general non-uniform lattice $x(s) = c_1q^s + c_2q^{-s} + c_3$ that such polynomials satisfy. The resulting expansion coefficients are given in a compact form in terms of the polynomial coefficients of the corresponding second-order difference equations. Notice that the above lattice contains, as a particular case, the exponential lattice $x(s) = q^s$ which was first considered in [4, 28]. The advantage of the present approach is that it only requires the knowledge of the second order difference equation satisfied by the involved hypergeometric $q$-polynomials as well as their hypergeometricity, i.e., the Rodrigues-type formula. Then, contrary to the algorithm presented in [3, 4, 28], we do not require information about any kind of recurrence relation of the involved discrete hypergeometric $q$-polynomials nor do we need to solve any high order recurrence relation for the connection coefficients themselves.

The structure of the paper is as follows. In Section 2, we collect the basic background [32] used in the rest of the work, namely, the second-order hypergeometric difference equation on the uniform lattice $x(s)$ and its polynomial solutions (called hypergeometric $q$-polynomials). In Section 3, we present the main results of the paper, namely, the expressions for the connection and linearization coefficients $c_{mn}$ and $c_{jmn}$, respectively. In particular, we show how the main formulas and theorems given in [5] for the linear lattice $x(s) = s$, as well as the ones given in [7, 41], hold as particular cases. Finally, in Section 4, some examples are developed.

2. SOME BASIC PROPERTIES OF THE $q$-POLYNOMIALS

Here we will summarize some of the properties of the $q$-polynomials [32] useful for the rest of the work.

2.1. The Hypergeometric-Type Difference Equation

Let us consider the second order difference equation of hypergeometric type

\[
\sigma(s) \frac{\Delta}{\Delta x(s-1/2)} \nabla y(s) + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda y(s) = 0,
\]

\[
\sigma(s) = \vartheta(x(s)) - \frac{1}{2} \bar{\vartheta}(x(s)) \Delta x(s - \frac{1}{2}), \quad \tau(s) = \bar{\vartheta}(x(s)), \quad (2.1)
\]

\[
\nabla f(s) = f(s) - f(s - 1), \quad \Delta f(s) = f(s + 1) - f(s),
\]
where $\nabla f(s)$ and $\Delta f(s)$ denote the backward and forward finite difference derivatives, respectively, $\overline{\sigma}(x)$ and $\overline{\tau}(x)$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and $\lambda$ is a constant. It is important to notice that the above difference equation has polynomial solutions of hypergeometric type iff $x(s)$ is a function of the form

$$x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_3(q) = c_1(q)[q^s + q^{-s-\mu}] + c_3(q),$$

(2.2)

where $c_1$, $c_2$, $c_3$, and $q^\mu = c_1/c_2$ are constants which, in general, depend on $q$ [12, 32, 33].

The polynomial solutions of (2.1) is determined by the $q$-analogue of the Rodrigues Formula on the non-uniform lattices [32, p. 66, Eq. (3.2.19)]

$$P_n(s)_q = \frac{B_n}{\rho(s)} \nabla^{(n)}[\rho_n(s)], \quad \nabla^{(n)} \equiv \frac{\nabla}{\nabla x_1(s)} \frac{\nabla}{\nabla x_2(s)} \cdots \frac{\nabla}{\nabla x_n(s)},$$

(2.3)

where the function $\rho_n(s)$ is given by

$$\rho_n(s) = \rho(s + n) \prod_{i=1}^{n} \sigma(s + i), \quad \text{and} \quad x_{\alpha n}(s) = x(s + \frac{n}{\alpha}),$$

(2.4)

and $\rho(s)$ is the solution of the Pearson-type difference equation

$$\frac{\Delta}{\Delta x(s - \frac{i}{\alpha})}[\sigma(s)\rho(s)] = \tau(s)\rho(s).$$

(2.5)

Throughout the paper $[n]_q$ denotes the so-called $q$-numbers and $[n]_q!$ are the $q$-factorial $[n]_q = (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$, $[n]_q! = [1]_q[2]_q \cdots [n]_q$.

Also the difference derivatives $y_kn(s)_q$ of the polynomial solution $P_n(s)_q$, defined by

$$y_kn(s)_q = \frac{\Delta}{\Delta x_{k-1}(s)} \frac{\Delta}{\Delta x_{k-2}(s)} \cdots \frac{\Delta}{\Delta x(s)}[P_n(s)_q] = \Delta^{(k)}[P_n(s)_q],$$

(2.6)

satisfy a Rodrigues-type formula

$$y_kn(s)_q = \Delta^{(k)}P_n(s)_q = \frac{A_{nk}B_n}{\rho_k(s)} \nabla^{(n)}[\rho_n(s)],$$

(2.7)
where the operator $\nabla_k^{(n)}$ is $\nabla_k^{(n)}f(s) = (\nabla/\nabla x_{k+1}(s))(\nabla/\nabla x_{k+2}(s)) \cdots (\nabla/\nabla x_n(s))[f(s)]$ and is given by (see Appendix A)

$$\nabla_k^{(n)}f(s) = \sum_{l=0}^{n-k} (-1)^l \frac{[n-k]_q!}{[l]_q![n-k-l]_q!} \times \frac{n}{\prod_{m=0}^{n-k} \nabla x_n(s - (m + l - 1/2))} f(s - l), \quad (2.8)$$

and

$$A_{nk} = \frac{[n]_q!}{[n-k]_q!} \frac{a_k}{B_k}, \quad (2.9)$$

where $a_n$ denotes the leading coefficient of the polynomial $P_n$, i.e., $P_n(x) = a_n x^n + \text{lower order terms}$.

In this paper we will deal with discrete orthogonal $q$-polynomials, i.e., polynomials with a discrete orthogonality

$$\sum_{x=a}^{b-1} P_n(x(s))q P_m(x(s))q \rho(s) \Delta x(s - 1/2) = \delta_{nm} d_n^2, \quad (2.10)$$

where $\rho(x)$ is a solution of the Pearson-type equation (2.5), and it is a nonnegative (not identically zero) weight function, i.e., $\rho(s) \Delta x(s - 1/2) \geq 0$, $a \leq s \leq b - 1$, supported on a countable subset of the real line $[a, b]$ ($a, b$ can be $\pm \infty$). This condition follows from the difference equation of hypergeometric-type (2.1), providing that the following boundary condition

$$\sigma'(s) \rho(s)x^k(s - 1/2)|_{s=a,b} = 0, \quad k = 0, 1, 2, \ldots, \quad (2.11)$$

holds [33]. Notice that the above boundary condition (2.11) is valid for $k = 0$. Moreover, if we assume that $a$ is finite, then (2.11) is fulfilled at $s = a$ providing that $\sigma(a) = 0$ [32, Sect. 3.3, p. 70]. In the following we will assume that this condition holds. The squared norm in (2.10) is given by [32, Chap. 3, Sect. 3.7.2, p. 104]

$$d_n^2 = (-1)^n A_{nn} B_n^2 \sum_{s=a}^{b-n-1} \rho_n(s) \Delta x_n(s - 1/2). \quad (2.12)$$
In the most general case, the solution of the \( q \)-hypergeometric equation (2.1) corresponds to the case

\[
\sigma(s) = A [s - s_1]_q [s - s_2]_q [s - s_3]_q [s - s_4]_q, \quad A = \text{const} \neq 0.
\]

and has the form [33]

\[
P_n(s)_q = B_n \left( \frac{-A}{c(q)q^{n\kappa_q}} \right)^n q^{-\frac{\gamma}{2}(s_1 + s_2 + s_3 + s_4 + \frac{\gamma q}{2})} \left( q^{s_1} ; q^\gamma \right)_n
\times \left( q^{s_1 + s_3 + \mu} ; q \right)_n \left( q^{s_1 + s_4 + \mu} ; q \right)_n \varphi_3
\times \left( q^{-n}, q^{\gamma + n - \frac{\gamma q}{2} + \sum_{i=1}^4 s_i - s}, q^{s_1 + s_3 + \mu}, q^{s_1 + s_4 + \mu} ; q, q \right),
\]

where \( \kappa_q = q^{1/2} - q^{-1/2} \), and the basic hypergeometric series \( \varphi_q \) are defined by [18]

\[
\varphi_p \left( a_1, a_2, \ldots, a_r ; b_1, b_2, \ldots, b_p ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1 ; q)_k \cdots (a_r ; q)_k}{(b_1 ; q)_k \cdots (b_p ; q)_k} z^k \left( (-1)^k q^{\frac{k(k-1)}{2}} \right)^{p - r + 1},
\]

and

\[
(a; q)_k = \prod_{m=0}^{k-1} (1 - aq^m)
\]

is a \( q \)-analogue of the Pochhammer symbol.

3. MAIN RESULTS

Here we find the explicit expression of the coefficients \( c_{mn} \) in the expansion of an arbitrary \( q \)-polynomial \( Q_m(s)_q \) on \( x(s) \) in series of the orthogonal discrete hypergeometric set of \( q \)-polynomials \( \{P_n\} \) in the same non-uniform lattice \( x(s) \), i.e.,

\[
Q_m(s)_q = \sum_{n=0}^{m} c_{mn} P_n(s)_q.
\]
**Theorem 3.1.** The explicit expression of the coefficients $c_{mn}$ in the expansion (3.1) is

$$c_{mn} = \frac{(-1)^n B_n}{d_n^2} \sum_{s=a}^{b-1} \Delta^{(n)}[Q_m(s)] \rho_n(s) \Delta x_n(s - \frac{1}{2})$$

$$= (-1)^n \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} \Delta^{(n)}[Q_m(s - n)] \rho_n(s - n) \Delta x(s - \frac{n+1}{2}). \quad (3.2)$$

**Proof.** Multiplying both sides of Eq. (3.1) by $P_k(s) \rho(x) \Delta x(s - \frac{1}{2})$, and summing between $a$ and $b - 1$, the orthogonality relation (2.10) immediately gives

$$c_{mn} = \frac{1}{d_n^2} \sum_{s=a}^{b-1} Q_m(s) P_n(s) \rho(s) \Delta x(s - \frac{1}{2}). \quad (3.3)$$

Next we use the Rodrigues formula (2.3) for $P_n(s)$. This yields

$$c_{mn} = \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} Q_m(s) \nabla^{(n)}[\rho_n(s)] \Delta x(s - \frac{1}{2})$$

$$= \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} Q_m(s) \nabla \left[ \nabla^{(n)}[\rho_n(s)] \right]. \quad (3.4)$$

Then, using the formula of summation by parts

$$\sum_{x_i=a}^{b-1} f(x_i) \nabla g(x_i) = f(x_i) g(x_i) \bigg|_{x_i=a}^{x_i=b-1} - \sum_{x_i=a}^{b-1} g(x_i - 1) \nabla f(x_i),$$

we obtain

$$c_{mn} = \frac{B_n}{d_n^2} Q_m(s) \nabla \left[ \nabla^{(n)}[\rho_n(s)] \right] \bigg|_{x_i=a}^{x_i=b-1}$$

$$- \frac{B_n}{d_n^2} \sum_{s=a}^{b-1} Q_m(s) \nabla \left[ \nabla^{(n)}[\rho_n(1)] \right] \bigg|_{s=1}^{s=s+1}. \quad (3.5)$$

Notice that the first term is proportional to $\rho_1(s) = \sigma(s + 1) \rho(s + 1)$, so, because of the condition (2.11), it vanishes. Now, making the change $s \to s - 1$ in the second term, we find

$$c_{mn} = - \frac{B_n}{d_n^2} \sum_{s=a}^{b-2} \Delta Q_m(s) \nabla \left[ \nabla^{(n)}[\rho_n(s)] \right].$$
But

\[ \nabla_1^{(n)}[\rho_n(s)] = \frac{\nabla}{\nabla x(s)} \nabla_2^{(n)}[\rho_n(s)], \quad \nabla x_2(s) = \nabla x(s + 1) = \Delta x(s). \]

Then the last equation transforms

\[ c_{mn} = -\frac{B_n}{d_n} \sum_{s=a-1}^{b-2} \frac{\Delta}{\Delta x(s)} [Q_m(s)_q] \nabla [\nabla_2^{(n)}[\rho_n(s)]] \]

Repeating this process \( k \) times, we obtain

\[ c_{mn} = (-1)^k \frac{B_n}{d_n} \sum_{s=a-k}^{b-k-1} \frac{\Delta}{\Delta x_{s-k}(s)} \cdots \frac{\Delta}{\Delta x(s)} [Q_m(s)_q] \nabla [\nabla_2^{(n)}[\rho_n(s)]] \]

Putting \( k = n \) and using \( \nabla^{(n)}[\rho_n(s)] = \rho_n(s) \) as well as that \( \rho_n(a - k) = 0 \) for \( k = 1, 2, \ldots, n \) (see Eq. (2.4)), we obtain the desired expression (3.2) for \( c_{mn} \).

To obtain the second expression for \( c_{mn} \) in (3.2) we use the identity

\[ \nabla_1^{(n)}[\rho_n(s - 1)] = \frac{\nabla}{\nabla x(s - 1)} \nabla_2^{(n)}[\rho_n(s - 1)] \]

\[ = \frac{\nabla}{\nabla x(s)} [\nabla_2^{(n)}[\rho_n(s - 1)]. \]

Then Eq. (3.5) becomes

\[ c_{mn} = -\frac{B_n}{d_n} \sum_{s=a}^{b-1} \frac{\nabla}{\nabla x(s)} [Q_m(s)_q] \nabla [\nabla_2^{(n)}[\rho_n(s)]] \]

Applying \( k \)-times the same technique as before we find

\[ c_{mn} = (-1)^k \frac{B_n}{d_n} \sum_{s=a}^{b-1} \frac{\nabla}{\nabla x(s - (k - 1)/2)} \frac{\nabla}{\nabla x(s - k/2 + 1)} \]

\[ \times \cdots \frac{\nabla}{\nabla x(s)} [Q_m(s)_q] \nabla_2^{(n)}[\rho_n(s - k)\Delta x(s - k + 1/2)]. \]
The change
\[ k = n \] and the fact that
\[ Q_n \]
lead us to the result.

**Corollary 3.1.** If \( Q_n \) is an hypergeometric polynomial satisfying an equation of the form (2.1) but with coefficients \( \tilde{\sigma}, \tilde{\tau}, \) and \( \tilde{\lambda}_m \), then the explicit expression of the coefficients \( c_{mn} \) in the expansion (3.1) is

\[
c_{mn} = \frac{(-1)^n B_n \tilde{B}_{mn} \tilde{A}_{mn}}{d_n^2} \sum_{l=0}^{m-n} (-1)^l \frac{[m-n]_q!}{[l]_q! [m-n-l]_q!} \times \sum_{s=a}^{b-n-1} \tilde{\rho}_m(s-l) \rho_n(s) \frac{\Delta x_m(s-l-1/2) \Delta x_n(s-l)}{\prod_{k=0}^{m-n} \Delta x_m(s-(k+l+1)/2)}.
\]

(3.6)

**Proof.** The proof follows from Eqs. (3.2), (2.6), and (2.8).

As a simple consequence of Theorem 3.1 we obtain the following result for the linearization problem, which consists of finding the expansion coefficients \( c_{jmn} \) of the relation

\[
R_j(s) Q_n(s) = \sum_{n=0}^{m+j} c_{jmn} P_n(s),
\]

where \( \{P_n\} \) is a discrete orthogonal set of hypergeometric \( q \)-polynomials which satisfy the difference equation (2.1) and \( Q_n \) and \( R_j \) are arbitrary \( q \)-polynomials on the same lattice \( x(s) \).

**Theorem 3.2.** The explicit expression of the coefficients \( c_{jmn} \) in the expansion (3.7) is

\[
c_{jmn} = \frac{(-1)^n B_n}{d_n^2} \sum_{s=a}^{b-n-1} \Delta^{(n)}[Q_n(s) R_j(s)] \rho_n(s) \Delta x_n(s - \frac{1}{2})
\]

(3.8)
or, equivalently,

\[
c_{jmn} = (-1)^n \frac{B_n}{d_n^2} \sum_{s=a}^{b-n-1} \frac{\nabla}{\nabla x(s-(n-1)/2)} \times \cdots \frac{\nabla}{\nabla x(s)} [Q_n(s) R_j(s)] \rho_n(s-n) \Delta x \left( s - \frac{n+1}{2} \right).
\]

(3.9)
THEOREM 3.3. Let $R_j$ be the $j$-degree $q$-hypergeometric polynomial solution of the second order difference equation on the non-uniform lattice $x(s)$

$$
\tilde{\alpha}(s) \frac{\Delta}{\Delta x(s-1/2)} \nabla y(s) + \tilde{\gamma}(s) \frac{\Delta y(s)}{\Delta x(s)} + \tilde{\lambda}_j y(s) = 0. \tag{3.10}
$$

Then the explicit expression of the coefficients $c_{jmn}$ in the expansion (3.7) is given by

$$
c_{jmn} = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=0}^n \frac{[n]_q!}{[k]_q! [n-k]_q!} \tilde{A}_{jk} \times \sum_{s=a}^{b-n-1} \frac{\rho_n(s) \Delta x_n(s-1/2)}{\tilde{\rho}_n(s+n-k)} \left[ \Delta^{(n-k)} Q_m(s)_q \left[ \nabla^{(j)} \tilde{p}_j(s+n-k) \right] \right],
$$

or, equivalently,

$$
c_{jmn} = \frac{(-1)^n B_n \tilde{B}_j}{d_n^2} \sum_{k=0}^n \frac{[n]_q!}{[k]_q! [n-k]_q!} \tilde{A}_{jn-k} \times \sum_{s=a}^{b-n-1} \frac{\rho_n(s) \Delta x_n(s-1/2)}{\tilde{\rho}_n(s+n-k)} \left[ \Delta^{(k)} Q_m(s+n-k)_q \left[ \nabla^{(j)} \tilde{p}_j(s) \right] \right].
$$

Proof. Using the $q$-analogue of the Leibniz formula in the non-uniform lattice (2.2) [1]

$$
\Delta^{(n)}[f(s)g(s)] = \sum_{k=0}^n \frac{[n]_q!}{[k]_q! [n-k]_q!} \Delta^{(k)}f(s+n-k) \Delta^{(n-k)}g(s),
$$

for the expression $\Delta^{(n)}[Q_m(s)_q R_j(s)_q]$ in Eq. (3.8), as well as the Rodrigues-type formula (2.6) for $\Delta^{(k)}R_j(s+n-k)_q$

$$
\Delta^{(k)}R_j(s+n-k)_q = \frac{\tilde{A}_{ik} \tilde{B}_j}{\tilde{\rho}_n(s+n-k)} \nabla^{(j)} \left[ \tilde{p}_j(s+n-k) \right],
$$

and the desired result holds. The second formula can be obtained analogously but starting from (3.9).
Corollary 3.2. The explicit expression of the coefficients $c_{jmn}$ in the expansion (3.7) is given by

$$c_{jmn} = \frac{(-1)^n B_n \hat{A}_j}{d_n^2} \sum_{k=0}^{b-n-1} \sum_{i=0}^{n-k} \frac{[n]_q!}{[k]_q! [n-k]_q!} \hat{A}_{jk} \sum_{l=0}^{j-k} (-1)^l \frac{[j-k]_q!}{[l]_q! [j-k-l]_q!} \times$$

$$\times \sum_{s=a}^{b-n-1} \frac{\rho_n(s) \Delta x_j(s - 1/2) \hat{p}_q(s + n - k - l)}{\hat{p}_k(s + n - k)} \times$$

$$\times \frac{\Delta x_j(s + n - k - l - 1/2)}{\prod_{m=0}^{j-k} \Delta x_j(s + n - k - (m + l + 1)/2)}$$

$$\times \Delta^{n-k} Q_m(s)_q.$$

(3.14)

Proof. To prove this it is sufficient to substitute the expression (2.8) in (3.11). \qed

Notice that Corollary 3.1 also follows from the above formula if we put $m = 0$ since $Q_0 = 1$.

3.1. Special Cases

3.1.1. The Classical Discrete Case

In the special case when $x(s)$ is the linear lattice, i.e., $x(s) = s$, from Theorems 3.1 and 3.2 we recover the main results in [5] for the connection and linearization problems, respectively.

Theorem 3.4. Let be $x(s)$ the linear lattice $x(s) = s$. Then the explicit expression of the coefficients $c_{m,n}$ in the expansion (3.1) is

$$c_{m,n} = \frac{(-1)^n B_n}{d_n^2} \sum_{s=a}^{b-n-1} \Delta^n Q_m(s) \rho(s + n) \prod_{k=1}^{n} \sigma(s + k)$$

$$= \frac{(-1)^n B_n}{d_n^2} \sum_{s=a}^{b-n-1} \Delta^n Q_m(s) \rho(s) \prod_{k=0}^{n-1} \sigma(s - k).$$

If $Q_m$ is also an hypergeometric polynomial, then

$$c_{m,n} = \frac{(-1)^n B_n \hat{A}_m}{d_n^2} \sum_{s=a}^{b-n-1} \sum_{k=0}^{m-n} \rho_n(s) \hat{p}_n(s) \binom{m-n}{k} \hat{p}_m(s-k)$$

$$= \frac{(-1)^n B_n \hat{A}_m}{d_n^2} \sum_{s=a}^{b-n-1} \sum_{k=0}^{m-n} \rho_n(s-n) \hat{p}_n(s-n) \binom{m-n}{k} \hat{p}_m(s-n-k)$$

$$\times \hat{p}_m(s-n-k).$$
Theorem 3.5. Let be $x(s)$ the linear lattice $x(s) = s$. Then the explicit expression of the coefficients $c_{jm}$ in the expansion (3.7) is given by

$$c_{jm} = \frac{(-1)^n B_nB_j}{d_n^2} \sum_{k=k_-}^{k_+} \binom{n}{k} A_{jk} \sum_{s=a}^{b-1} \sum_{l=0}^{j-k} (-1)^l \binom{j-k}{l}$$

$$\times \frac{\rho_a(s)}{\tilde{\rho}_x(s + n - k)} \tilde{\rho}_x(s + n - k + l) [\nabla^{n-k} Q_m(s + n - k)]$$

$$= \frac{(-1)^n B_nB_j}{d_n^2} \sum_{k=k_-}^{k_+} \binom{n}{k} A_{jk} \sum_{s=a}^{b-1} \sum_{l=0}^{j-k} (-1)^l \binom{j-k}{l}$$

$$\times \frac{\rho_a(s - n)}{\tilde{\rho}_x(s - k)} \tilde{\rho}_x(s - k + l) [\nabla^{n-k} Q_m(s - k)],$$

where $k_- = \max(0, n - m)$ and $k_+ = \min(n, j)$.

3.1.2. The Classical Continuous Case

Finally, we will show how from Theorem 3.1 we can recover (formally) the general results for the continuous case [7, 41]. In order to do this we notice that, formally, if we make the change $x(s) = sh \rightarrow x$, then [32]

$$P_n(x(s + 1)) - P_n(x(s)) \frac{x(s + 1) - x(s)}{h} = P_n(sh + h) - P_n(sh)$$

$$= \frac{P_n(x + h) - P_n(x)}{h}.$$ 

Thus, $\lim_{h \to 0} (\Delta P_n(x(s))/\Delta x_k(s)) = P'_n(x)$ and $\lim_{h \to 0} \Delta^{(k)} P_n(s) = d^k P_n(s)/dx^k$. Then by similar limiting processes Eq. (2.1) transforms into the classical hypergeometric differential equation [32]

$$\sigma(x) P'_n(x) + \tau(x) P_n(x) + \lambda_n P_n(x) = 0,$$

where $\sigma(x) = \lim_{s \to 0} \sigma(x(s))$, $\tau(x) = \lim_{s \to 0} \tau(x(s))$ being $x = sh$. Furthermore, the Pearson-type equation (2.5) becomes $[\sigma(x)\rho(x)]' = \tau(x)\rho(x)$ and also [32] $\rho_a(s; h) \to \rho(x)\sigma^n(x)$. Finally, the Rodrigues-type formula (2.3) transforms into

$$\Delta^{(k)} P_n(s)_i = \frac{A_{nk}B_n}{\rho_k(s)} \nabla^k[ \rho_n(s)] \rightarrow \frac{d^k P_n(x)}{dx^k}$$

$$= \frac{A_{nk}B_n}{\rho_k(x)} \frac{d^{n-k}}{dx^{n-k}} [ \rho(x)\sigma^n(x)].$$
Now we put $x(s) = sh$ in (3.2)

$$c_{mn}(h) = \frac{(-1)^n B_n(h)}{d^2_n(h)} \sum_{x_i = ah}^{B - nh} \Delta^{(n)}[Q_m(x_i)_q] \rho_n(x_i/h; h)h$$

$$= \frac{(-1)^n B_n(h)}{d^2_n(h)} \sum_{x_i = A}^{B - nh} \Delta^{(n)}[Q_m(x_i)_q] \rho_n(x_i/h; h)h,$$

$$x_{i+1} = x_i + h.$$

Let us prove that the above sum transforms in the limit in an integral from which the main result in [41, Theorem 3.1, p. 163] easily follows. More concretely,

$$\lim_{h \to 0} c_{mn}(h) = \left(\frac{(-1)^n B_n}{d^2_n} \int_A^B \frac{d^k Q_m(x)}{dx^k} \rho(x) \sigma^n(x) \, dx\right),$$

where $d^2_n$ is the squared norm for the polynomials orthogonal with respect to $\rho(x)$ [32].

In order to do that, let us show that the quantity

$$I_n(Q_m, \rho_n) = \left| \sum_{x_i = A}^{B - nh} \Delta^{(n)}[Q_m(sh)_q] \rho_n(x_i/h; h)h - \int_A^B Q_m^{(n)}(x) \rho(x) \sigma^n(x) \, dx \right|$$

can be small enough for $h$ sufficiently small,

$$|I_n(Q_m, \rho_n)| \leq \sum_{x_i = A}^{B - nh} \left| \Delta^{(n)}[Q_m(sh)_q] - Q_m^{(n)}(x_i)_q \right| \rho_n(x_i/h; h)h$$

$$+ \sum_{x_i = A}^{B - nh} \left| Q_m^{(n)}(x_i) \rho_n(x_i/h; h) - \rho_n(x_i) \right| h$$

$$+ \int_A^B Q_m^{(n)}(x) \rho_n(x) \, dx,$$

where $Q_m^{(n)}$ denotes the $n$th derivative of $Q_m$ and $\rho_n(x) = \rho(x)\sigma^n(x)$. Let consider first the case when $B$ is bounded. In this case the first integral can be small enough (less than $\epsilon/3$) for $h$ sufficiently small providing that $\rho_n(x_i/h; h)$ is bounded. In the following we will suppose that the limit function $\rho_n(x)$, $n \geq 1$, is a continuous function in $[A, B]$. For the second
sum we can do the same since $Q_m$ is a polynomial and then it is bounded in any closed interval. Finally we will consider the last sum which can be rewritten in the form

$$\left| \sum_{x_i=A}^{B} Q_m^{(n)}(x_i) \rho_n(x_i) h - \int_{A}^{B} Q_m^{(n)}(x) \rho_n(x) \, dx \right| + \left| \sum_{x_i=B-hn}^{B} Q_m^{(n)}(x_i) \rho_n(x_i) h \right|.$$

Notice that the first sum can be less than $\epsilon/6$ since it is a Riemann sum corresponding to the integral $\int_{A}^{B} Q_m^{(n)}(x) \rho_n(x) \, dx$, and the last sum obviously tends to zero so, for sufficiently small $h$, it is less than $\epsilon/6$. So, for any given $\epsilon > 0$, one can choose a sufficiently small $h$ so that $|I_n(Q_m, \rho_n)| \leq \epsilon$.

Finally, to prove the result for the unbounded $B$ we use the fact that, in this case, the functions $\rho_n(x_i/h; h)$ as well as $\rho_n(x_i)$ tend to zero faster than any polynomial tends to infinity when $x_i \to \infty$ (see the boundary conditions (2.11) for the polynomials on the lattice $x(s)$ as well as for the continuous case [32, Eq. (1.3.1), p. 7]). Then

$$|I_n(Q_m, \rho_n)| \leq \sum_{x_i=A}^{\infty} |\Delta^{(n)} \left[ Q_m(s h) \right] - Q_m^{(n)}(x_i) \rho_n(x_i/h; h) h + \sum_{x_i=A}^{\infty} [Q_m^{(n)}(x_i) \{ \rho_n(x_i/h; h) - \rho_n(x_i) \}] | h$$

$$+ \left| \sum_{x_i=A}^{\infty} Q_m^{(n)}(x_i) \rho_n(x_i) h - \int_{A}^{\infty} Q_m^{(n)}(x) \rho_n(x) \, dx \right| \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

To conclude this section let us point out that here we have taken the limit formally and have proved that our main result, i.e., formula (3.2), transforms into the corresponding one for the continuous case [41]; but for solving concrete examples one must be very careful since, for instance, in the limit Hahn $\rightarrow$ Jacobi, the parameter $h = 1/N$ where $N$ is the total number of points in the lattice and the Hahn polynomials explicitly depend
on it. More information on how to take limits for concrete families can be found in [18, 23, 32, 33].

4. EXAMPLES

4.1. Connection between \((q^s; q)_m\) and \(c_n^\mu(s, q)\)

First of all we will apply Theorem 3.1 for finding the connection coefficients \(c_n^\mu\) in the expansion

\[
(q^s; q)_m = \sum_{n=0}^{m} c_n^q c_n^\mu(s, q),
\]

where \((a; q)_k\) is defined in (2.16), and \(c_n^\mu(s, q)\) denotes the \(q\)-Charlier polynomials on the exponential lattice \(x(s) = (q^s - 1)/(q - 1) [2]\)

\[
c_n^{(\mu)}(s, q) = q^{2(n+\frac{\mu}{\mu})} \Phi_0 \left( q^{-n} q^{-s} q; \frac{q^s}{(q - 1) \mu} \right)
= q^{2(n+\frac{\mu}{\mu})} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k \mu^k} (s)_q^{[k]}, \quad 1 < q < 1, 0 < \mu < 1.
\]

Obviously, these \(q\)-Charlier polynomials \(c_n^{(\mu)}(s, q)\) are polynomials of degree \(n\) on any exponential lattice \(x(s) = c_1 q^s + c_3\). For these polynomials we have [2]

\[
\rho(s) = \frac{\mu^s}{c_q \left[ (1 - q) \mu \right] \Gamma_q(s + 1)},
\]
\[
a_n = \frac{(-1)^n}{\mu^n q^{\frac{n}{2}(n-1)+\frac{s}{2}}}, \quad B_n = \frac{1}{\mu^n},
\]
\[
d_n^2 = \frac{e_q \left[ (1 - q) q^{n+1} \mu \right]}{e_q \left[ (1 - q) \mu \right] q^{\frac{n(s-9)}{2} + \frac{s}{2}}} \frac{[n]_q!}{\mu^n},
\]
\[
\rho_n(s) = \frac{\mu^s n^{2(n+2s+1)}}{e_q \left[ (1 - q) \mu \right] \Gamma_q(s + 1)},
\]
where \( e_q[z] \) denotes the \( q \)-exponential function \([18]\) defined by

\[
e_q[z] = \sum_{k=0}^{\infty} \frac{z^k}{(q^k q)_k} = \frac{1}{(z; q)_\infty}, \quad \Gamma_q(s)
\]

\[
= \left(1 - q\right)^{1-s} \prod_{k=0}^{\infty} \frac{(1 - q^{k+1})}{(1 - q^{s+k})}, \quad \text{and} \quad (z; q)_\infty = \prod_{k=0}^{\infty} (1 - zq^k).
\]

(4.5)

Notice that all the characteristics given in (4.3) and (4.4), as well as the polynomials \( c_n^{(\mu)}(s, q) \) themselves, transform into the classical Charlier polynomials in the limit \( q \to 1 \). Notice also that the results presented here remain valid for the \( q \)-Charlier polynomials in the lattice \( x(s) = q^s [1, 2] \) since their hypergeometric representation is given by the same basic hypergeometric series (4.2).

Now, using

\[
\frac{\Delta(q^s; q)_n}{\Delta x(s)} = -q^{\frac{s}{2} - \frac{n}{2}} \left[ q^{s-1} (q^{s+1}; q)_{n-1} - (q^s; q)_n \right], \quad x(s) = c_1 q^s + c_3,
\]

we get

\[
\Delta^n[(q^s; q)_m] = q^{-\frac{s(n-1)}{2}} \left[ \frac{\Delta}{\Delta x(s)} \right]^n (q^s; q)_m
\]

\[
= \frac{(1 - q)^n [m! q^{s(m-1)}]}{[m-n]_q!} (q^{s+n}; q)_{m-n}.
\]

Therefore (3.2) gives

\[
c_{m,n}^q = \frac{q^{\frac{s(5n-7)}{2}} (q - 1) \mu^n m! q^{n+1}}{e_q \left( (1 - q) \mu q^{n+1} \right)} \prod_{s=0}^{\infty} \frac{(q^{s+n}; q)_{m-n} (1 - q^{s+1})}{(q; q)_s},
\]

where the \( q \)-binomial coefficients are defined by \( \binom{m}{n}_q = (q; q)_m/(q; q)_n(q; q)_{m-n} \).

In order to take the sum in the above expression we will use the identity \([18, \text{Eq. (1.2.34), p. 6}] \) \( (aq^s; q)_n = (a; q)_n (aq^k; q)/(q; q)_n \), as well as the expression \([18, \text{Eq. (1.5.2), p. 11}] \) \( (a^m; q)_n/(q^m; q)_n = \sum_{k=0}^{n} ((q^{m-k}; q)_k (q^m; q)_k)/(q^m; q)_k/(q; q)_k) \). Thus, denoting by
\( z = (1 - q) \mu q^{n+1} \), we have

\[
\sum_{s=0}^{\infty} \frac{(q^{n+m}; q)_{m-n} z^s}{(q; q)_s} = \sum_{s=0}^{\infty} \frac{(q^n; q)_{m-n}(q^m; q)_s z^s}{(q; q)_s}
\]

\[
= (q^n; q)_{m-n} \sum_{k=0}^{\infty} \frac{(q^{n-m}; q)_k q^{mk} z^k}{(q^n; q)_k(q; q)_k} \sum_{s=0}^{\infty} \frac{(q^{-s}; q)_k q^{sk} z^s}{(q; q)_s}
\]

\[
= (q^n; q)_{m-n} \sum_{k=0}^{\infty} \frac{(q^{n-m}; q)_k q^{mk} z^k}{(q^n; q)_k(q; q)_k} \left[ (1 - q)^k \frac{q^{k(k-1)} - 1}{q} \right] \sum_{s=k}^{\infty} \frac{z^{s-k}}{(q; q)_{s-k}}
\]

\[
= (q^n; q)_{m-n} e_q \left[ (1 - q) \mu q^{n+1} \right] \varphi_1 \left( \frac{q^{n-m}}{q^n}; q, \mu q^{n+1} (1 - q) \right).
\]

For the third equality we have used the identity [18, Eq. (1.2.32), p. 6]

\[
\frac{(q^{-s}; q)_k}{(q; q)_s} = \frac{(-1)^k q^{s(k-1) - ks}}{(q; q)_{s-k}}.
\]

Then for the coefficients \( c_{mn}^q \) we finally obtain

\[
c_{mn}^q = (q^n; q)_{m-n} \mu^n (q - 1)^n q^{2(n-1)} \binom{m}{n} q^{\frac{n-m}{q}}
\]

\[
\times \varphi_1 \left( \frac{q^{n-m}}{q^n}; q, \mu q^{n+1} (1 - q) \right).
\]

**Remark.** Notice that, since \( (q^k; q)_m / (1 - q)^m = \sum_{n=0}^{m} (c_{mn}^q)/(1 - q)^m c_{n}^m(x, q) \), and taking into account that \( \lim_{q \to 1} ((q^n; q)_m / (1 - q)^m) = (s)_m \), \( \lim_{q \to 1} c_{n}^m(x, q) = c_{n}^m(s) \), we obtain taking the limit \( q \to 1 \)

\[
(s)_m = \sum_{n=0}^{m} c_{mn} c_{n}^m(s), \quad c_{mn} = \binom{m}{n} \frac{(m-1)!}{(n-1)!} (-\mu)_n \varphi_1 \left( \frac{n-m}{n}; -\mu \right).
\]

where \( c_{n}^m(s) \) denotes the classical (non-monic) Charlier polynomials [32, 33]. Since for these polynomials the leading coefficients are given by \( a_m = (-\mu)^{-n} \), the above result coincides with the classical result (see, e.g., [5] and references therein).
4.2. Connection between \((q^s; q)^{[m]}\) and \(c_n^{\mu}(x, q)\)

Now we will apply Theorem 3.1 for finding the connection coefficients \(c_n^{\mu}\) in the expansion

\[
(q^s; q)^{[m]} = \sum_{n=0}^{m} d_{mn}^{\mu} c_n^{\mu}(s, q),
\]

(4.9)

where \((a; q)^{[k]}\) is given by

\[
(s)_{q}^{[n]} = \frac{\Gamma_q(s) \Gamma_q(n)}{\Gamma_q(s+n)},
\]

(4.10)

and \(c_n^{\mu}(s, q)\) is, as before, the \(q\)-Charlier polynomials on the lattice

\[
x(s) = c_0 q^s + c_3,
\]

(4.11)

we obtain

\[
\Delta^n \left[ (q^s; q)^{[m]} \right] = q^{-\frac{n}{2}(n-1)} \left[ \frac{\Delta}{\Delta x(s)} \right]^n (q^s; q)^{[m]}
\]

\[
= \frac{(1 - q)^n [m]_q q^{-\frac{n}{2}(m-1)}}{[m-n]_q} (q^{s+n}; q)_{m-n}.
\]

Then, using formula (3.2), the expression \((q^s; q)^{[m-n]}(q; q)_s = 1/(q; q)_{s-m+n},\) as well as

\[
\sum_{s=0}^{\infty} \frac{(q^s; q)^{[m-n]} z^s}{(q; q)_s} = \sum_{s=m-n}^{\infty} \frac{(q^s; q)^{[m-n]} z^s}{(q; q)_s}
\]

\[
= z^{m-n} \sum_{s=0}^{\infty} \frac{z^s}{(q; q)_s} = z^{m-n} e_q(z),
\]

we obtain

\[
d_{mn}^{\mu} = q^{m+\frac{\mu}{2}(n-\gamma)} \binom{m}{n}_q (1 - q)^m (-1)^\gamma \mu^m.
\]

(4.12)

The above formula is the \(q\)-analogue of the so-called inversion formula for hypergeometric polynomials (compare with the explicit expression of the \(q\)-Charlier polynomials (4.2)).
Remark. If we rewrite (4.9) in the form

\[ (s)_q^{[m]} = \sum_{n=0}^{m} \tilde{d}_{mn}^q \mu_n^m(s, q), \quad \tilde{d}_{mn}^q = q^{m + \frac{n}{2}(n-1)} \binom{m}{n}_q (-1)^n \mu_n^m. \] (4.13)

taking into account that \( \lim_{q \to 1} (q^s; q)_q^{[m]} / (1 - q)^m = (s)_q^{[m]} \), we obtain in the limit \( q \to 1 \)

\[ (s)_q^{[m]} = \sum_{n=0}^{m} d_{mn} \mu_n^m(s), \quad d_{mn} = \binom{m}{n} (-1)^n \mu_n^m. \]

Using again the fact that for the polynomials \( c_n^m(s) \), the leading coefficients are given by \( a_n = (-\mu)^n \), the above result coincides with the well known classical result (see, e.g., [5] and references therein).

4.3. The \( q \)-Charlier Polynomials in the Exponential Lattice

Finally, we will solve now the connection problem

\[ c_n^q(s, q) = \sum_{n=0}^{m} c_{mn}^q \mu_n^m(s, q). \] (4.14)

Then by using the expression (3.6) of Corollary 3.1 where \( Q_n(s)_q = c_n^q(s, q) \) and \( P_n(s)_q = c_n^\mu(s, q) \), respectively, we obtain

\[
c_{mn}^q = \left( \frac{\mu}{q^q} \right) \binom{m}{n}_q q^{l(m - n)(m - n + 5)} \sum_{l=0}^{m-n} \frac{(-1)^l q^{2l(2m-l-1)}}{(1 - q)^l \gamma^l} \binom{m - n}{l}_q
\]

\[
\times \sum_{s=l}^{\infty} \frac{[(1 - q) \mu q^{n + 1}]^{l-s}}{(q; q)_{l-s}}
\]

\[
= \left( \frac{\mu}{q^q} \right) \binom{m}{n}_q q^{l(m - n)(m - n + 5)} \sum_{l=0}^{m-n} \frac{(-1)^l \left( \frac{\mu}{q} q^{n + 1} \right)^l \binom{m - n}{l}_q}{(1 - q)^l \gamma^l}
\]

where we also use the fact that

\[
\sum_{s=0}^{\infty} \frac{z^k}{\Gamma_q(s - k)} = \sum_{s=k}^{\infty} \frac{z^k}{\Gamma_q(s - k)} = \sum_{s=k}^{\infty} \frac{z^k (1 - q)^s - k}{(q; q)_s - k}.
\]

Now, applying the identity (4.7) to \( (q; q)_{m-n-l}^{-1} (k = l) \), and using the \( q \)-binomial theorem [18, Sect. 1.3, Eq. (1.3.14), p. 9],

\[
\sum_{l=0}^{k} \binom{q^{-k}; q}_l z^l = \frac{1}{\varphi_q \left( \frac{q^{-k}; q}{z}; q, z \right)} = (zq^{-k}; q)_k,
\]
we obtain the following expression for the coefficient $c_{m,n}^{q}$

$$c_{m,n}^{q} = \binom{\mu}{\gamma} m^n q^{\frac{n}{2}(m-n)(m-n+5)}(q^{n-m+1} + \mu \gamma^{-1}; q)_{m-n}. \quad (4.15)$$

Notice the positivity of the coefficients (4.15) in the case when $\mu/\gamma < q^{m-1}$. In the case of the Charlier polynomials in the exponential lattice a similar result has been obtained in [28] by solving a recurrence relation for the coefficients (there the author does not give a closed formula for the connection coefficient).

**Remark.** A simple calculation shows that Eq. (4.14) transforms in the limit $q \to 0$ into

$$c_{m}^{q}(s) = \sum_{n=0}^{m} \binom{m}{n} \left( \frac{\mu}{\gamma} \right)^n \left( 1 - \frac{\mu}{\gamma} \right)^{m-n} c_{n}^{q}(s),$$

for the (non-monic) Charlier polynomials and this coincides with the classical results for monic polynomials (see, e.g., [5]) since the leading coefficients for the Charlier polynomials $c_{n}^{q}(s)$ are equal to $(-\mu)^{-n}$.

### 4.4. Further Examples

To conclude the paper we will show two more examples for polynomials on $q$-quadratic lattices, more exactly in the lattice $x(s) = [s]_q[s + 1]_q$, i.e., $c_1 = q^{1/2} \gamma^{-2}$ and $\mu = 1$. In this case there are no structure relations and then most of the aforementioned (in the Introduction) method cannot be used.

In fact we will solve the two examples,

$$(q^{-s}; q)_m(q^{s+1}; q)_m = \sum_{n=0}^{m} d_{mn} u_{n}^{\alpha, \beta}(x, 0, b), \quad \text{and}$$

$$u_{n}^{\gamma, \delta}(x, 0, d) = \sum_{n=0}^{m} c_{mn} u_{n}^{\alpha, \beta}(x, 0, b),$$

where $u_{n}^{\alpha, \beta}(x, 0, b)$ denotes the $q$-Racah polynomials introduced by Nikiforov and Uvarov in [33] (see also [1])

$$u_{n}^{\alpha, \beta}(x, 0, b) = \varphi_{3}\left( q^{-n}, q^{\alpha + \beta + n + 1}, q^{-s}, q^{s+1}; q^{-b+1}, q^{b+1}, q^{b+\alpha+1}; q, q \right). \quad (4.16)$$
For these polynomials we have

\[ \rho(s) = \frac{q^{1-s}(1 - q^{-1})(1 - q^{-2})\Gamma(s + \beta + 1)\Gamma(s + \alpha + b + 1)\Gamma(b + \alpha - s)}{\Gamma(s + b + 1)\Gamma(s - \beta + 1)\Gamma(b - s)} , \]

\[ d_n^2 = q^{-\frac{\alpha b + a \beta + 2n}{2}} \times \frac{\Gamma(\alpha + n + 1)\Gamma(\beta + n + 1)\Gamma(\alpha + \beta + n + 1)\Gamma(b + \alpha + n + 1)}{[\alpha + \beta + 2n + 1]_{n}\Gamma(\alpha + \beta + n + 1)\Gamma(b - n)\Gamma(b - \beta - n)} . \]

In the first case, using the identity

\[ \frac{\Delta}{\Delta x(s)} (q^{1-s}; q)_m (q^{1+s+\mu}; q)_n = -q^{2s+1+n} \frac{[k]}{[k]} c_{-1}^{-1} (q^{1-s}; q)_m (q^{1+s+\mu+1}; q)_n , \]

for the lattice \( x(s) = c_1(q)[q^{\mu} + q^{-\mu}] + c_3(q) \), and (3.2) we obtain

\[ d_{mn} = \frac{(m)}{(n)} q^{-\frac{\alpha b + a \beta + 2n}{2}} \frac{(q^{-b+1}; q)_m (q^{b+1}; q)_n (q^{b+n+1}; q)_n}{(q^{a+\beta+n+1}; q)_n (q^{a+b+2n+1}; q)_n} . \]

Finally, using (3.6), after some straightforward but cumbersome calculations we find

\[ c_{mn} = (-1)^n q^{-\frac{\alpha b + a \beta + 2n}{2}} \frac{(q^{-m}; q)_n (q^{a+\beta+n+m+1}; q)_n (q^{-d+1}; q)_n (q^{d+1}; q)_n (q^{d+n+1}; q)_n}{(q; q)_n (q^{-b+1}; q)_n (q^{b+1}; q)_n (q^{b+n+1}; q)_n (q^{b+n+1}; q)_n} \times \phi_2 \left( \begin{array}{c} q^{n-m}q^{a+\beta+n+m+1}q^{n-d+1}q^{d+n+1}q^{d+\gamma+n+1} \\ q^{\gamma+\delta+2n+1}q^{\gamma+n+1}q^{\gamma+n+1}q^{\gamma+n+1}q^{\gamma+n+1} \\ q ; q \end{array} \right) . \]

(4.17)

Notice that if we assume that \( q \in (0, 1) \) and take the limit \( \gamma \to \infty \) we obtain the connection between \( q \)-Racah and \( q \)-Dual Hahn \( W_n^{(\delta)}(x(s), 0, d)_q = \phi_2 \left( \begin{array}{c} q^{n-m}q^{a+\beta+n+m+1}q^{n-d+1}q^{d+n+1}q^{d+n+1} \\ q^{\gamma+\delta+2n+1}q^{\gamma+n+1}q^{\gamma+n+1}q^{\gamma+n+1}q^{\gamma+n+1} \\ q ; q \end{array} \right) [1] \)

\[ c_{mn} = (-1)^n q^{-\frac{\alpha b + a \beta + 2n}{2}} \frac{(q^{-m}; q)_n (q^{a+\beta+n+m+1}; q)_n (q^{-d+1}; q)_n (q^{d+1}; q)_n (q^{d+n+1}; q)_n}{(q; q)_n (q^{-b+1}; q)_n (q^{b+1}; q)_n (q^{b+n+1}; q)_n (q^{b+n+1}; q)_n} \times \phi_3 \left( \begin{array}{c} q^{n-m}q^{a+\beta+n+m+1}q^{n-d+1}q^{d+n+1}q^{d+n+1} \\ q^{\gamma+\delta+2n+1}q^{\gamma+n+1}q^{\gamma+n+1}q^{\gamma+n+1}q^{\gamma+n+1} \\ q^{\gamma+\delta+n+1}q^{\gamma+n+1}q^{\gamma+n+1}q^{\gamma+n+1}q^{\gamma+n+1} \\ q ; q \end{array} \right) . \]
From the above equation follows, by taking the limits $\alpha, \gamma \to \infty$, a formula for the connection coefficients for the $q$-Dual Hahn–$q$-Dual Hahn polynomials.

APPENDIX A

In this appendix we will prove the expression (2.8). In fact we will prove the following lemma which is interesting in its own right:

**Lemma.** Let $f(s)$ be an analytic function inside and on a curve $C$ on the complex plane containing the points $z = s, s-1, \ldots, s-n$, and $\nabla_k^{(n)}$ the operator

$$\nabla_k^{(n)} = \frac{\nabla}{\nabla x_{k+1}(s)} \frac{\nabla}{\nabla x_{k+2}(s)} \cdots \frac{\nabla}{\nabla x_n(s)}.$$

Then

$$\nabla_k^{(n)} f(s) = \sum_{l=0}^{n-k} (-1)^l \frac{[n-k]_q!}{[l]_q! [n-k-l]_q!} \nabla x_n(s-l+1/2) f(s-l), \quad (A.1)$$

Proof: First of all, notice that the function $x_m(z) = x(z + \frac{m}{2})$, where $x(s)$ is given by (2.2), satisfies

$$x(s) - x(s-t) = [t]_q x(s - \frac{t-1}{2}). \quad (A.2)$$

Then, by induction, one has

$$\nabla_k^{(n)} \left[ \frac{1}{x_n(z) - x_n(s)} \right] = \frac{[n-k]_q! \prod_{m=0}^{k} [x_n(z) - x_n(s-m)]}{\prod_{m=0}^{k} [x_n(z) - x_n(s-m)]}$$

$$= \frac{[n-k]_q!}{[x_n(z) - x_n(s)]^{(n-k+1)}},$$

where

$$[x_k(z) - x_k(s)]^{(m)} = \prod_{j=0}^{m-1} [x_k(z) - x_k(s-j)], \quad m = 0, 1, 2, \ldots,$$

$(A.3)$
denotes the generalized powers. Since $f$ is analytic, then by using the Cauchy formula

$$f(s) = \frac{1}{2\pi i} \int_C \frac{f(z) x_n'(z)}{x_n(z) - x_n(s)} \, dz,$$

we have

$$f(s) = \frac{n-k}{2\pi i} \int_C \frac{f(z) x_n'(z)}{x_n(z) - x_n(s)} \, dz.$$  \hspace{1cm} (A.5)$$

If we now use the residue's theorem, and taking into account that the only singularities of the integrand are the simple poles located at $z = s - l$, $l = 0, 1, \ldots, n - k$, then

$$Res \left[ \frac{f(z) x_n'(z)}{[x_n(z) - x_n(s)]^{n-k+1}} \right] = \frac{f(s-l)}{\prod_{m=0}^{n-k} [x_n(s-l) - x_n(s-m)]}.$$  

Finally, using the property (A.2) the result follows.

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