Distortion of boundary sets under inner functions (II)

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Version date: July 9, 1993
1. Introduction.

An inner function is a bounded holomorphic function from the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ into the unit disk $\Delta$ of the complex plane such that the radial boundary values have modulus 1 almost everywhere. If $E$ is a non empty Borel subset of $\partial \Delta$, we denote by $f^{-1}(E)$ the following subset of the unit sphere $\mathbb{S}_n$ of $\mathbb{C}^n$

$$f^{-1}(E) = \{ \xi \in \mathbb{S}_n : \lim_{r \to 1} f(r \xi) \text{ exist and belongs to } E \}.$$ 

The classical lemma of Löwner, see e.g. [R, p.405], asserts that inner functions $f$, with $f(0) = 0$, are measure preserving transformations when viewed as mappings from $\mathbb{S}_n$ to $\partial \Delta$, i.e. if $E$ is a Borel subset of $\partial \Delta$ then $|f^{-1}(E)| = |E|$, where in each case $| \cdot |$ means the corresponding normalized Lebesgue measure.

In this paper we extend this result to fractional dimensions as follows:

**Theorem 1.** If $f$ is inner in the unit disk $\Delta$, $f(0) = 0$, and $E$ is a Borel subset of $\partial \Delta$, we have:

$$\text{cap}_\alpha(f^{-1}(E)) \geq \text{cap}_\alpha(E), \quad 0 \leq \alpha < 1.$$ 

Moreover, if $E$ is any Borel subset of $\partial \Delta$ with $\text{cap}_\alpha(E) > 0$, equality holds if and only if either $f$ is a rotation or $\text{cap}_\alpha(E) = \text{cap}_\alpha(\partial \Delta)$.

Moreover, it is well known, see [N], that if $f$ is not a rotation then $f$ is ergodic, i.e., there are no nontrivial sets $A$, with $f^{-1}A = A$ except for a set of Lebesgue measure zero. This also has a fractional dimensional parallel.

**Corollary.** With the hypotheses of Theorem 1, if $f$ is not a rotation and if the symmetric difference between $E$ and $f^{-1}(E)$ has zero $\alpha$-capacity, then either $\text{cap}_\alpha(E) = 0$ or $\text{cap}_\alpha(E) = \text{cap}_\alpha(\partial \Delta)$.

**Theorem 2.** If $f$ is inner in the unit ball of $\mathbb{C}^n$, $f(0) = 0$, and $E$ is a Borel subset of $\partial \Delta$, we have:

$$\text{cap}_{2n-2+\alpha}(f^{-1}(E)) \geq K(n, \alpha)^{-1} \text{cap}_\alpha(E), \quad 0 < \alpha < 1,$$

and

$$\frac{1}{\text{cap}_{2n-2}(f^{-1}(E))} \leq 1 + (2n - 2) \log \frac{1}{\text{cap}_0(E)}, \quad (n > 1).$$

**Corollary.** In particular, for any inner function $f$, we have that

$$\text{Dim}(f^{-1}(E)) \geq 2n - 2 + \text{Dim}(E),$$

where Dim denotes Hausdorff dimension.
Here $\text{cap}_\alpha$ and $\text{cap}_0$ denote, respectively, $\alpha$-dimensional Riesz capacity and logarithmic capacity. We refer to [C], [KS] and [L] for definitions and basic background on capacity.

For background and some applications of this results we refer to [FP] where it is shown that Theorem 1 holds with some constants depending on $\alpha$.

The outline of this paper is as follows: In Section 2 we obtain an integral expression for the $\alpha$-energy that is used in Section 3, where theorems 1 and 2 are proved. Section 4 contains some further results for the case $n = 1$. In Section 5, we prove an analogous distortion theorem, with Hausdorff measures replacing capacities. Section 6 contains an open question.

We would like to thank José Galé and Francisco Ruiz-Blasco for some helpful conversations concerning the energy functional. Also, we would like to thank David Hamilton for suggesting that the right constant in Theorem 1 is 1 (see [H]).

2. An integral expression for the $\alpha$-energy.

In this section we obtain an expression of the $\alpha$-energy of a signed measure $\mu$ in $\Sigma_{N-1}$ (the unit sphere of $\mathbb{R}^N$) as an $L^2$-norm of its Poisson extension. This approach is due to Beurling [B].

If $\mu$ is a signed measure on $\Sigma_{N-1}$, and $0 \leq \alpha < N-1$, then the $\alpha$-energy $I_\alpha(\mu)$ of $\mu$ is defined as

$$I_\alpha(\mu) = \iint_{\Sigma_{N-1} \times \Sigma_{N-1}} \Phi_\alpha(|x-y|) \, d\mu(x) \, d\mu(y),$$

where

$$\Phi_\alpha(t) = \begin{cases} \log \frac{1}{t}, & \text{if } \alpha = 0, \\ \frac{1}{t^\alpha}, & \text{if } 0 < \alpha < N-1. \end{cases}$$

Recall that if $E$ is a closed subset of $\Sigma_{N-1}$, then

$$(\text{cap}_\alpha(E))^{-1} = \inf \{I_\alpha(\mu) : \mu \text{ probability supported on } E\},$$

for $0 < \alpha < N-1$,

$$\log \frac{1}{\text{cap}_0(E)} = \inf \{I_0(\mu) : \mu \text{ probability supported on } E\},$$

and that the infimum is attained by a unique probability $\mu_c$ which is called the equilibrium distribution on $E$. 

If $E$ is any Borel subset of $\Sigma_{N-1}$, then the $\alpha$-capacity of $E$ is defined as
\[
\text{cap}_\alpha(E) = \sup\{\text{cap}_\alpha(K) : K \subset E, \ K \text{ compact}\}.
\]
We recall Choquet’s theorem that all Borel sets are \textit{capacitables}, \textit{i.e.}
\[
\text{cap}_\alpha(E) = \inf\{\text{cap}_\alpha(O) : E \subset O, \ O \text{ open}\}.
\]
As we shall remark later on, for a general Borel set $E$ of $\Sigma_{N-1}$, one has
\[
\frac{1}{\text{cap}_\alpha(E)} = \inf\{I_\alpha(\mu) : \mu \text{ probability, } \mu(E) = 1\},
\]
and analogously for the logarithmic capacity.

We first need to obtain the expansion of the integral kernel $\Phi_\alpha$ in terms of the spherical harmonics. We refer to [SW, chap.IV] for details about spherical harmonics; we shall follow their notations.

Let $\mathcal{H}_k$ be the real vector space of the spherical harmonics of degree $k$ in $\mathbb{R}^N$ ($N > 1$). If $a_k$ is the dimension of $\mathcal{H}_k$, we have
\[
a_0 = 1, \quad a_1 = N, \quad a_k = \frac{N + 2k - 2}{k} \binom{N + k - 3}{k - 1} \quad \text{[SW, p.145].}
\]
If $\Sigma_{N-1}$ denotes the unit sphere of $\mathbb{R}^N$, the space $L^2(\Sigma_{N-1}, d\xi)$ can be decomposed as
\[
L^2(\Sigma_{N-1}, d\xi) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k,
\]
where $d\xi$ is the usual Lebesgue measure (not normalized).

If $\xi, \eta$ belongs to $\Sigma_{N-1}$, $Z^k_\eta(\xi)$ will denote the zonal harmonic of degree $k$ with pole $\eta$, and if $\{Y^k_1, \ldots, Y^k_{a_k}\}$ is any orthonormal basis of $\mathcal{H}_k$, we have
\[
Z^k_\eta(\xi) = \sum_{m=1}^{a_k} Y^k_m(\xi)Y^k_m(\eta) = Z^k_\xi(\eta) \quad \text{[SW, p.143].}
\]
The zonal harmonics can be expressed in terms of the ultraspherical (or Gegenbauer) polynomials $P^\lambda_k$ which are defined by the formula
\[
(1 - 2rt + r^2)^{-\lambda} = \sum_{k=0}^{\infty} P^\lambda_k(t)r^k,
\]
where $|r| < 1$, $|t| \leq 1$ and $\lambda > 0$. 
We have [SW, p.149], if $N > 2$,
\[ Z^k_\eta(\xi) = C_{k,N} P^{(N-2)/2}_k(\xi \cdot \eta). \]

It is easy to compute the constants $C_{k,N}$. First, if $\omega_{N-1}$ denotes the Lebesgue measure of $\Sigma_{N-1}$, then
\[ \|Z^k_\eta\|_2 = \frac{a_k}{\omega_{N-1}} \]
while, on the other hand,
\[ \frac{a_k}{\omega_{N-1}} = C_{k,N} \int_{\Sigma_{N-1}} |P^{(N-2)/2}_k(\xi \cdot \eta)|^2 d\xi \]
\[ = C_{k,N} \omega_{N-2} \int_{-1}^1 |P^{(N-2)/2}_k(t)|^2 (1 - t^2)^{(N-3)/2} dt. \]

Now, the polynomials $P^{(N-2)/2}_k(t)$ form an orthogonal basis of
\[ L^2([-1, 1], (1 - t^2)^{(N-3)/2} dt) \]
[SW, p.151], [AS, p.774], and
\[ \|P^{(N-2)/2}_k\|_2^2 = \frac{\pi 2^{4-N} \Gamma(k + N - 2)}{k! (2k + N - 2) \Gamma\left(\frac{N - 2}{2}\right)^2} \]
[AS, p.774],
where $\Gamma(\cdot)$ denotes the Euler’s Gamma function, and, therefore
\[ C_{k,N}^2 = \frac{a_k}{\omega_{N-1} \omega_{N-2}} \|P^{(N-2)/2}_k\|_2^{-2} = \frac{(N + 2k - 2)^2}{16 \pi^N} \Gamma\left(\frac{N - 2}{2}\right)^2. \]

Hence
\[ C_{k,N} = \frac{N + 2k - 2}{4 \pi^{N/2} \Gamma\left(\frac{N - 2}{2}\right)}, \]
and
\[ Z^k_\eta(\xi) = \frac{N + 2k - 2}{4 \pi^{N/2} \Gamma\left(\frac{N - 2}{2}\right)} P^{(N-2)/2}_k(\xi \cdot \eta). \]

The case $N = 2$ is slightly different. In this case we can take $P^0_k = T_k$, the Chebyshev’s polynomials defined in $[-1, 1]$ by
\[ T_k(\cos \theta) = \cos k\theta. \]

It is known that these polynomials form an orthogonal basis of
\[ L^2([-1, 1], (1 - t^2)^{-1/2} dt). \]
In this particular case, if \( \xi = e^{i\theta}, \eta = e^{i\psi}, \) then \( \xi \cdot \eta = \cos(\theta - \psi), \) and
\[
Z^k_\eta(\xi) = \frac{1}{\pi} \cos k(\theta - \psi) = \frac{1}{\pi} T_k(\cos(\theta - \psi)) = \frac{1}{\pi} P^0_k(\xi \cdot \eta), \quad k = 1, 2, \ldots ,
\]
\[
Z^0_\eta(\xi) = \frac{1}{2\pi} = \frac{1}{2\pi} P^0(\xi \cdot \eta).
\]
Therefore,
\[
C_{k,2} = \begin{cases} 
\frac{1}{\pi}, & \text{if } k > 0, \\
\frac{1}{2\pi}, & \text{if } k = 0.
\end{cases}
\]

We can now write down the expansion of the kernel \( \Phi_\alpha(|x - y|) \) in a Fourier series of Gegenbauer’s polynomials. Fix, first, \( \alpha, \) with \( 0 < \alpha < N - 1. \) If we denote by \( g(t) \) the function
\[
g(t) = \left( \frac{1}{2 - 2t} \right)^{\alpha/2},
\]
then we can express the kernel \( \Phi_\alpha \) in terms of \( g \) as
\[
\Phi_\alpha(|\xi - \eta|) = \Phi_\alpha(\sqrt{|\xi|^2 - 2\xi \cdot \eta + |\eta|^2}) = g(\xi \cdot \eta).
\]
Now, develop \( g(t) \) as a Fourier series
\[
g(t) = \sum_{k=0}^{\infty} g_k P^{(N-2)/2}_k(t), \quad \text{where} \quad g_k \|P^{(N-2)/2}_k\|_2^2 = \langle g, P^{(N-2)/2}_k \rangle,
\]
and conclude
\[
(1) \quad \Phi_\alpha(|\xi - \eta|) = g(\xi \cdot \eta) = \sum_{k=0}^{\infty} g^k Z^k_\eta(\xi),
\]
where \( g^k C_{k,N} = g_k. \)

Hereafter \( F \) will denote the usual hypergeometric function
\[
F(a, b; c; t) = \sum_{m=0}^{\infty} \frac{(a)_m(b)_m t^m}{(c)_m m!},
\]
where
\[
(u)_m = u(u + 1) \ldots (u + m - 1) = \frac{\Gamma(u + m)}{\Gamma(u)}.
\]
The polynomials \( P^{(N-2)/2}_k \) can be expressed in terms of \( F \) [AS, p.779].
If \( N > 2 \),
\[
P_k^{(N-2)/2}(t) = \binom{k + N - 3}{k} F(-k, k + N - 2; (N - 1)/2; (1 - t)/2).
\]

Then,
\[
\langle g, P_k^{(N-2)/2} \rangle = \binom{k + N - 3}{k} \int_{-1}^{1} s^{-1+(N-1-\alpha)/2} (1 - s)^{-1+(N-1)/2} \cdot (2 - 2t)^{-\alpha/2} (1 - t^2)^{(N-3)/2} \, dt.
\]

Therefore
\[
\langle g, P_k^{(N-2)/2} \rangle = 2^{N-2-\alpha} \binom{k + N - 3}{k} \int_{0}^{1} s^{-1+(N-1-\alpha)/2} (1 - s)^{-1+(N-1)/2} \cdot F(-k, k + N - 2; (N - 1)/2; 1 - s) \, ds.
\]

Using the relationship
\[
P_k^{(N-2)/2}(-t) = (-1)^k P_k^{(N-2)/2}(t), \quad \text{[SW, p.149], [AS, p.775]},
\]
we have
\[
\langle g, P_k^{(N-2)/2} \rangle = 2^{N-2-\alpha} \binom{k + N - 3}{k} (-1)^k \int_{0}^{1} s^{-1+(N-1-\alpha)/2} (1 - s)^{-1+(N-1)/2} \cdot F(-k, k + N - 2; (N - 1)/2; 1 - s) \, ds.
\]

Term by term integration of the series defining \( F \) gives
\[
\int_{0}^{1} s^{a-1}(1 - s)^{b-1} F(-k, c; b; 1 - s) \, ds = \beta(a, b; 1 - s) \, ds = \beta(a, b; 1 - s),
\]
where \( \beta(\cdot, \cdot) \) is the Euler’s Beta function.

Moreover, it is easy to see that ([AS, p.556])
\[
F(-k, c; a + b; 1) = \frac{\Gamma(a + b) \Gamma(a + b - c + k)}{\Gamma(a + b + k) \Gamma(a + b - c)}
\]
\[
= \frac{\Gamma(a + b)}{\Gamma(a + b + k)} (-1)^k \frac{\Gamma(1 + c - a - b)}{\Gamma(1 + c - a - b - k)} ,
\]
and so
\[
(-1)^k \int_{0}^{1} s^{a-1}(1 - s)^{b-1} F(-k, c; b; 1 - s) \, ds = \frac{\Gamma(a) \Gamma(b) \Gamma(1 + c - a - b)}{\Gamma(a + b + k) \Gamma(1 + c - a - b - k)} .
\]
This gives

\[ \langle g, P_k^{(N-2)/2} \rangle = 2^{N-\alpha} \binom{k + N - 3}{k} \frac{\Gamma \left( \frac{N - 1 - \alpha}{2} \right) \Gamma \left( \frac{N - 1}{2} \right) \Gamma \left( k + \frac{\alpha}{2} \right)}{\Gamma \left( N - 1 - \frac{\alpha}{2} + k \right) \Gamma \left( \frac{\alpha}{2} \right)} , \]

and

\[ g_k = \frac{\langle g, P_k^{(N-2)/2} \rangle}{\| P_k^{(N-2)/2} \|^2} = 2^{N-3-\alpha} \frac{N + 2k - 2}{\sqrt{\pi}} \frac{\Gamma \left( \frac{N - 1 - \alpha}{2} \right) \Gamma \left( \frac{N}{2} - 1 \right) \Gamma \left( k + \frac{\alpha}{2} \right)}{\Gamma \left( N - 1 - \frac{\alpha}{2} + k \right) \Gamma \left( \frac{\alpha}{2} \right)} . \]

Therefore,

(2) \[ g^k = g_k C_{k,N}^{-1} = 2^{N-1-\alpha} \pi^{(N-1)/2} \frac{\Gamma \left( \frac{N - 1 - \alpha}{2} \right) \Gamma \left( k + \frac{\alpha}{2} \right)}{\Gamma \left( N - 1 - \frac{\alpha}{2} + k \right) \Gamma \left( \frac{\alpha}{2} \right)} , \]

if \( N > 2 \). On the other hand, if \( N = 2 \), the \( k \)-th Chebyshev’s polynomial is \( T_k(t) = F(-k, k; 1/2; (1-t)/2) \), (see [AS, p.779]), and

\[ \langle g, P_k^0 \rangle = \int_{-1}^{1} (2 - 2t)^{-\alpha/2} F(-k, k; 1/2; (1-t)/2)(1-t^2)^{-1/2} dt . \]

Using the above computations when \( N = 2 \), we have that

\[ \langle g, P_k^0 \rangle = 2^{-\alpha} \pi^{1/2} \frac{\Gamma \left( \frac{1 - \alpha}{2} \right) \Gamma \left( k + \frac{\alpha}{2} \right)}{\Gamma \left( 1 - \frac{\alpha}{2} + k \right) \Gamma \left( \frac{\alpha}{2} \right)} . \]

Moreover it is easy to see, [AS, p.774], that

\[ \| P_k^0 \|^2 = \begin{cases} \frac{\pi}{2}, & \text{if } k > 0, \\ \frac{\pi}{2}, & \text{if } k = 0, \end{cases} \]

and also that \( C_{k,2}^{-1} = 2 \| P_k^0 \|^2 \).

Then

\[ g^k = \frac{\langle g, P_k^0 \rangle}{\| P_k^0 \|^2} C_{k,2}^{-1} , \]

and so (2) is also satisfied in this case \( (N = 2) \). Therefore we have proved the following:
Lemma 1. For all $N \in \mathbb{N}$, $N > 1$ and $0 < \alpha < N - 1$, 
\[
\Phi_\alpha(|\xi - \eta|) = \sum_{k=0}^{\infty} g^k Z_\eta^k(\xi),
\]
where
\[
g^k = 2^{N-1-\alpha} \pi^{(N-1)/2} \frac{\Gamma \left( \frac{N-1-\alpha}{2} \right) \Gamma \left( k + \frac{\alpha}{2} \right)}{\Gamma \left( \frac{N-1-\alpha}{2} + k \right) \Gamma \left( \frac{\alpha}{2} \right)}.
\]

Now we can express the $\alpha$-energy of a measure $\mu$ in terms of its Poisson extension $P_\mu$.

Lemma 2. If $\mu$ is a signed measure supported on $\Sigma_{N-1}$, we have:

(i) If $0 < \alpha < N - 1$, then
\[
I_\alpha(\mu) = C(N, \alpha) \int_0^1 \left\{ \int_{\Sigma_{N-1}} |P_\mu(r\xi)|^2 d\xi \right\} r^{\alpha-1} (1 - r^2)^{N-2-\alpha} dr,
\]
with
\[
C(N, \alpha) = \frac{4\pi^{N/2}}{\Gamma \left( \frac{\alpha}{2} \right) \Gamma \left( \frac{N-\alpha}{2} \right)}.
\]

(ii) If $m = \mu(\Sigma_{N-1})$, then
\[
I_0(\mu) = \omega_{N-1} \int_0^1 \int_{\Sigma_{N-1}} \left| P_\mu(r\xi) - \frac{m}{\omega_{N-1}} \right|^2 r^{\alpha-1} (1 - r^2)^{N-2-\alpha} dr d\xi
\]
\[
+ \frac{m^2}{2} \left[ \frac{\Gamma'}{\Gamma \left( \frac{N}{2} \right)} - \frac{\Gamma'}{\Gamma(N-1)} \right].
\]

In particular, if $N = 2$,
\[
I_0(\mu) = 2\pi \int_0^1 \int_0^{2\pi} \left| P_\mu(re^{i\theta}) - \frac{m}{2\pi} \right|^2 d\theta dr.
\]

Proof. Let $\{\mu_j^k\}$, $k \geq 0$, $1 \leq j \leq a_k$, be the Fourier coefficients of $\mu$, i.e.,
\[
\mu \sim \sum_{k=0}^{\infty} \sum_{j=1}^{a_k} \mu_j^k \gamma_j^k.
\]
Recall that $P_\mu$ is defined by
\[
P_\mu(r\xi) = \int_{\Sigma_{N-1}} p(\eta, r\xi) d\mu(\eta),
\]
where \( p(\eta, r \xi) \) is the classical (normalized) Poisson kernel
\[
p(\eta, r \xi) = \frac{1}{\omega_{N-1}} \frac{1-r^2}{|\eta-r \xi|^N}.
\]

We have [SW, p.145]
\[
p(\eta, r \xi) = \sum_{k=0}^{\infty} r^k Z^k_{\eta}(\xi) = \sum_{k,j} r^k Y^k_j(\eta) Y^k_j(\xi).
\]

Now, Plancherel's theorem gives
\[
P_\mu(r \xi) = \sum_{k,j} r^k \mu^k_j Y^k_j(\xi).
\]

Using again Plancherel's theorem we obtain
\[
\int_{\Sigma_{N-1}} |P_\mu(r \xi)|^2 \, d\xi = \sum_{k,j} r^{2k} |\mu^k_j|^2,
\]
and so if we define \( \Lambda \) as the right hand side in (i), we have
\[
\Lambda = C(N, \alpha) \sum_{k,j} |\mu^k_j|^2 \int_0^1 r^{2k+\alpha-1} (1-r^2)^{N-2-\alpha} \, dr,
\]
and, substituting \( r^2 = t \), we get that
\[
\Lambda = \frac{C(N, \alpha)}{2} \sum_{k,j} \frac{\Gamma\left(k + \frac{\alpha}{2}\right) \Gamma(N-1-\alpha)}{\Gamma\left(k + N-1 - \frac{\alpha}{2}\right)} |\mu^k_j|^2 = \sum_{j,k} g^k |\mu^k_j|^2.
\]

Note that we have used the known duplication formula for the Gamma function in the last equality.

On the other hand, by (1),
\[
\Phi_\alpha(|\xi - \eta|) = \sum_{k=0}^{\infty} g^k Z^k_{\eta}(\xi) = \sum_{k,j} g^k Y^k_j(\eta) Y^k_j(\xi),
\]
and using Plancherel's theorem we obtain
\[
\int_{\Sigma_{N-1}} \Phi_\alpha(|\xi - \eta|) \, d\mu(\eta) = \sum_{k,j} g^k \mu^k_j Y^k_j(\xi),
\]
\[
I_\alpha(\mu) = \sum_{k,j} g^k |\mu^k_j|^2 = \Lambda.
\]
This finishes the proof of (i).

In order to prove (ii) observe that

$$\int_{\Sigma N-1} \left| P_\mu(r\xi) - \frac{m}{\omega_{N-1}} \right|^2 d\xi + \frac{m^2}{\omega_{N-1}} = \int_{\Sigma N-1} \left| P_\mu(r\xi) \right|^2 d\xi.$$

Integrating this equality we have

$$I_\alpha(\mu) = C(N, \alpha) \int_0^1 \int_{\Sigma N-1} \left| P_\mu(r\xi) - \frac{m}{\omega_{N-1}} \right|^2 d\xi r^{\alpha-1}(1-r^2)^{N-2-\alpha} dr$$

$$+ m^2 U(\alpha),$$

where

$$U(\alpha) = \frac{\Gamma(N/2)\Gamma(N-1-\alpha)}{\Gamma((N-\alpha)/2)\Gamma(N-1-\alpha/2)},$$

and hence

$$\lim_{\alpha \to 0} \frac{I_\alpha(\mu) - m^2 U(\alpha)}{\alpha} = \omega_{N-1} \int_0^1 \int_{\Sigma N-1} \left| P_\mu(r\xi) - \frac{m}{\omega_{N-1}} \right|^2 d\xi (1-r^2)^{N-2} dr / r.$$ 

On the other hand,

$$\lim_{\alpha \to 0} \frac{I_\alpha(\mu) - m^2 U(\alpha)}{\alpha} = \lim_{\alpha \to 0} \frac{I_\alpha(\mu) - m^2}{\alpha} - \frac{m^2}{\omega_{N-1}} \lim_{\alpha \to 0} \frac{U(\alpha) - 1}{\alpha} = I_0(\mu) - m^2 U'(0),$$

and

$$U'(0) = \frac{1}{2} \left[ \frac{\Gamma'(N/2)}{\Gamma(N/2)} - \frac{\Gamma'(N-1)}{\Gamma(N-1)} \right].$$

This finishes the proof of Lemma 2. \qed

3. Distortion of $\alpha$-capacity.

We need the following lemmas.

Lemma 3. Let $\mu$ be a finite positive measure in $\partial \Delta$, and let $f$ be an inner function. Then, there exists a unique positive measure $\bar{\nu}$ in $\mathbb{S}_n$ such that $P_\mu \circ f = P_{\bar{\nu}}$ and

$$\bar{\nu}(f^{-1}(\text{support } \mu)) = \tilde{\nu}(\mathbb{S}_n).$$

Therefore if, moreover, $f(0) = 0$, then

$$\frac{1}{\omega_{2n-1}} \tilde{\nu}(\mathbb{S}_n) = \frac{1}{2\pi} \mu(\partial \Delta).$$
Proof. It is essentially the same as Lemma 1 in [FP], but see Lemma 14 below for further details.

A different normalization is useful; choosing \( \nu = (2\pi / \omega_{2n-1}) \bar{\nu} \), one obtains

\[
P_\nu = \frac{2\pi}{\omega_{2n-1}} P_\mu \circ f \quad \text{and} \quad \nu(\mathbb{S}_n) = \mu(\partial \Delta).
\]

The following is well known

**Lemma 4.** (Subordination principle) Let \( f : \mathbb{B}_n \rightarrow \Delta \) be a holomorphic function such that \( f(0) = 0 \), and let \( \nu : \Delta \rightarrow \mathbb{R} \) be a subharmonic function. Then

\[
\frac{1}{\omega_{2n-1}} \int_{\mathbb{S}_n} \nu(f(r \xi)) \, d\xi \leq \frac{1}{2\pi} \int_0^{2\pi} \nu(re^{i\theta}) \, d\theta.
\]

It will be relevant later on to recall the well known fact that, in the case \( n = 1 \), equality in Lemma 4 holds if and only if either \( \nu \) is harmonic or \( f \) is a rotation. Note also that there is no such equality statement when \( n > 1 \), since in higher dimensions the extremal functions in Schwarz’s lemma are not so clearly determined (see e.g. [R, p.164]).

**Lemma 5.** Let \( \mu \) be a signed measure on \( \partial \Delta \), \( f \) an inner function with \( f(0) = 0 \), and \( \nu \) a signed measure on \( \mathbb{S}_n \) such that

\[
P_\nu = (2\pi / \omega_{2n-1}) P_\mu \circ f.
\]

Then

(i) If \( n = 1 \) and \( 0 \leq \alpha < 1 \), then

\[
I_\alpha(\nu) \leq I_\alpha(\mu).
\]

(ii) If \( n > 1 \) and \( 0 < \alpha < 1 \), then

\[
I_{2n-2+\alpha}(\nu) \leq K(n, \alpha) I_\alpha(\mu),
\]

where

\[
K(n, \alpha) = \frac{(n-1)! \Gamma \left( \frac{\alpha}{2} \right)}{\Gamma \left( n - 1 + \frac{\alpha}{2} \right)}.
\]

If \( \alpha = 0 \) and \( m = \mu(\partial \Delta) = \nu(\mathbb{S}_n) \), we have

\[
I_{2n-2}(\nu) \leq (2n - 2) I_0(\mu) + m^2.
\]
The measure \( \nu \) is obtained from Lemma 3 by splitting \( \mu \) into its positive and negative parts. Note that for fixed \( \alpha \),

\[
K(n, \alpha) \sim n^{1-\alpha/2} \Gamma \left( \frac{\alpha}{2} \right), \quad \text{as } n \to \infty,
\]
while for fixed \( n > 1 \)

\[
K(n, \alpha) \sim \frac{C_n}{\alpha}, \quad \text{as } \alpha \to 0.
\]

Let us observe also that \( K(n, \alpha) \) takes the value 1 for \( n = 1 \).

**Proof.** Since \( |P_\mu - \frac{m}{2\pi}|^2 \) and \( |P_\mu|^2 \) are subharmonic, we obtain by subordination, Lemma 4, that if \( n = 1 \) and \( \alpha = 0 \)

\[
\int_0^{2\pi} \left| P_\nu - \frac{m}{2\pi} \right|^2 d\theta = \int_0^{2\pi} \left| P_\mu(f) - \frac{m}{2\pi} \right|^2 d\theta \leq \int_0^{2\pi} \left| P_\mu - \frac{m}{2\pi} \right|^2 d\theta,
\]

and if \( n \geq 1, 0 < \alpha < 1 \), that

\[
(3) \quad \int_{\mathbb{S}_n} |P_\nu|^2 d\xi = \left( \frac{2\pi}{\omega_{2n-1}} \right)^2 \int_{\mathbb{S}_n} |P_\mu(f)|^2 d\xi \leq \frac{2\pi}{\omega_{2n-1}} \int_0^{2\pi} |P_\mu|^2 d\theta.
\]

In the first case, we obtain

\[
I_0(\nu) \leq I_0(\mu)
\]

by integrating with respect to \( 2\pi dr/r \) and applying Lemma 2, part (ii).

In the second case, using Lemma 2, part (i), and Lemma 4 with \( v = |P_\mu|^2 \), we have that

\[
I_{2n-2+\alpha}(\nu) = C(2n, 2n - 2 + \alpha) \int_0^1 \left\{ \int_{\mathbb{S}_n} |P_\nu(r\xi)|^2 d\xi \right\} r^{2n-2+\alpha-1} \frac{dr}{(1-r^2)^\alpha} \leq \frac{C(2n, 2n - 2 + \alpha)}{C(2, \alpha)} \cdot \frac{2\pi}{\omega_{2n-1}} \int_0^1 \left\{ \int_0^{2\pi} |P_\mu(r e^{i\theta})|^2 d\theta \right\} r^{\alpha-1} \frac{dr}{(1-r^2)^\alpha} = K(n, \alpha) I_{\alpha}(\mu),
\]

where

\[
K(n, \alpha) = \frac{(n-1)! \Gamma \left( \frac{\alpha}{2} \right)}{\Gamma(n-1 + \frac{\alpha}{2})}.
\]

Finally, since \( \nu(\mathbb{S}_n) = m \),

\[
\int_{\mathbb{S}_n} \left| P_\nu(r\xi) - \frac{m}{\omega_{2n-1}} \right|^2 d\xi = \int_{\mathbb{S}_n} |P_\nu(r\xi)|^2 d\xi - \frac{m^2}{\omega_{2n-1}},
\]
and so, Lemma 2 gives, if \( n > 1 \),
\[
I_{2n-2}(\nu) = m^2 + \frac{4\pi^n}{(n-2)!} \int_0^1 \int_{S_n} \left| P_\nu(r\xi) - \frac{m}{\omega_{2n-1}} \right|^2 \, d\xi \, r^{2n-3} \, dr .
\]
By lemmas 3 and 4, we get
\[
\int_{S_n} \left| P_\nu(r\xi) - \frac{m}{\omega_{2n-1}} \right|^2 \, d\xi = \int_{S_n} \left| \frac{2\pi}{\omega_{2n-1}} P_\mu(f(r\xi)) - \frac{m}{\omega_{2n-1}} \right|^2 \, d\xi
= \left( \frac{2\pi}{\omega_{2n-1}} \right)^2 \int_{S_n} \left| P_\mu(f(r\xi)) - \frac{m}{2\pi} \right|^2 \, d\xi
\leq \frac{2\pi}{\omega_{2n-1}} \int_0^{2\pi} \left| P_\mu(re^{i\theta}) - \frac{m}{2\pi} \right|^2 \, d\theta .
\]
Therefore,
\[
I_{2n-2}(\nu) \leq m^2 + \frac{4\pi^n}{(n-2)!} \int_0^1 \frac{2\pi}{\omega_{2n-1}} \int_0^{2\pi} \left| P_\mu(re^{i\theta}) - \frac{m}{2\pi} \right|^2 \, d\theta \, \frac{dr}{r}
= m^2 + \frac{4\pi^n}{(n-2)!} \frac{1}{\omega_{2n-1}} I_0(\mu)
= m^2 + (2n-2)I_0(\mu).}

The proof of Lemma 5 is finished. \( \square \)

Finally, we can prove

**Theorem 1.** If \( f \) is inner in the unit disk \( \Delta \), \( f(0) = 0 \), and \( E \) is a Borel subset of \( \partial \Delta \), we have:
\[
\text{cap}_\alpha(f^{-1}(E)) \geq \text{cap}_\alpha(E), \quad 0 \leq \alpha < 1 .
\]
Moreover, if \( E \) is any Borel subset of \( \partial \Delta \) with \( \text{cap}_\alpha(E) > 0 \), equality holds if and only if either \( f \) is a rotation or \( \text{cap}_\alpha(E) = \text{cap}_\alpha(\partial \Delta) \).

Notice the following consequence concerning invariant sets. It is well known that an inner function \( f \) with \( f(0) = 0 \), which is not a rotation, is ergodic with respect to Lebesgue measure, see e.g. [P]. As a consequence of the above, it is also ergodic with respect to \( \alpha \)-capacity. More precisely,

**Corollary.** With the hypotheses of Theorem 1, if \( f \) is not a rotation and if the symmetric difference between \( E \) and \( f^{-1}(E) \) has zero \( \alpha \)-capacity, then either \( \text{cap}_\alpha(E) = 0 \) or \( \text{cap}_\alpha(E) = \text{cap}_\alpha(\partial \Delta) \).

In higher dimensions we have
Theorem 2. If \( f \) is inner in the unit ball of \( \mathbb{C}^n \), \( f(0) = 0 \), and \( E \) is a Borel subset of \( \partial \Delta \), we have:

\[
\text{cap}_{2n-2+\alpha}(f^{-1}(E)) \geq K(n, \alpha)^{-1} \text{cap}_{\alpha}(E), \quad 0 < \alpha < 1,
\]

and

\[
\frac{1}{\text{cap}_{2n-2}(f^{-1}(E))} \leq 1 + (2n - 2) \log \frac{1}{\text{cap}_0(E)}, \quad (n > 1).
\]

Proof of Theorems 1 and 2. To prove the inequalities in the theorems we may assume that \( E \) is closed. Assume first that \( n = 1, 0 < \alpha < 1 \). Let us denote by \( \mu_e \) the \( \alpha \)-equilibrium probability distribution of \( E \), and let \( \nu \) be the probability measure such that \( P_\nu = P_{\mu_e} \circ f \). By Lemma 5,

\begin{equation}
I_\alpha(\nu) \leq I_\alpha(\mu_e) = (\text{cap}_\alpha(E))^{-1}.
\end{equation}

But, from Lemma 3, \( I_\alpha(f^{-1}(E)) = 1 \), and so

\[
I_\alpha(\nu) = \int_{f^{-1}(E)} \Phi_{\alpha}(|z - \nu|) \, d\nu(z) \, d\nu(w).
\]

Now, let \( \{K_n\} \) be an increasing sequence of compacts subsets in \( \partial \Delta \), \( K_n \subset f^{-1}(E) \), such that \( \nu(K_n) \to 1 \). The monotone convergence theorem gives

\begin{equation}
I_\alpha(\nu) \geq \lim_{n \to \infty} (\text{cap}_\alpha(K_n))^{-1} = \inf_n (\text{cap}_\alpha(K_n))^{-1}
\end{equation}

\[
\geq (\text{cap}_\alpha(f^{-1}(E)))^{-1}.
\]

The inequality in Theorem 1 follows now from (4) and (5).

The cases \( n > 1 \) (Theorem 2) and \( n = 1, \alpha = 0 \) are completely analogous.

Proof of Equality Statement in Theorem 1. We prove it first in the case that \( E \) is closed. We include it since its proof shows more clearly the ideas that we will use to demonstrate the general case.

Suppose that \( 0 < \alpha < 1 \). We have seen that

\[
\frac{1}{\text{cap}_\alpha(f^{-1}(E))} \leq I_\alpha(\nu) \leq I_\alpha(\mu_e) = \frac{1}{\text{cap}_\alpha(E)}.
\]

Therefore, if \( E \) and \( f^{-1}(E) \) have the same \( \alpha \)-capacity, then

\[
I_\alpha(\nu) = I_\alpha(\mu_e),
\]

and this is possible only if for all \( r \in (0, 1) \),

\[
\int_0^{2\pi} |P_{\mu_e}(re^{i\theta})|^2 \, d\theta = \int_0^{2\pi} |P_{\mu_e}(f(re^{i\theta}))|^2 \, d\theta.
\]
This can occur only if either \( f \) is a rotation or \(|P_{\mu_e}|^2\) is harmonic. In the latter case, we obtain that \( \mu_e \) is normalized Lebesgue measure, or equivalently that \( \text{cap}_\alpha(E) = \text{cap}_\alpha(\partial \Delta) \). Since \( E \) is closed, it follows that \( E = \partial \Delta \).

In order to prove the general case we need a characterization of the \( \alpha \)-capacity of \( E \) when \( E \) is not closed (see Lemmas 10 and 10' below). We begin by recalling some facts about convergence of measures.

We will say that a sequence of signed measures \( \{\sigma_n\} \) with supports contained in a compact set \( K \) converges \( w^* \) to a signed measure \( \sigma \) if

\[
\int h(x) \, d\sigma_n(x) \xrightarrow{n \to \infty} \int h(x) \, d\sigma(x), \quad \text{for all } h \in C(K).
\]

Here, the \( w^* \)-convergence makes reference to the duality between the space of signed measures on \( K \) and the space \( C(K) \) of continuous functions with support contained in \( K \).

In this Section, we will denote by \( \mathcal{M}_\alpha(K) \) \((0 \leq \alpha < 1)\) the vector space of all signed measures with support contained in \( K \) and whose \( \alpha \)-energy is finite. \( \mathcal{M}_\alpha \) will be the same space without restriction on the support of the measure, and \( \mathcal{M}_\alpha^+ \) the corresponding space of positive measures.

**Remark 1.** If \( \{\sigma_n\} \subset \mathcal{M}_\alpha^+ \), \( 0 \leq \alpha < 1 \) and \( \sigma_n \xrightarrow{w^*} \sigma \), then \( \sigma \in \mathcal{M}_\alpha^+ \) and

\[
I_\alpha(\sigma) \leq \liminf_{n \to \infty} I_\alpha(\sigma_n).
\]

**Proof of Remark 1.** Let \( \Phi_{\alpha, \rho}(t) \) be the “truncated” kernel

\[
\Phi_{\alpha, \rho}(t) = \begin{cases} 
\Phi_{\alpha}(t), & \text{if } \Phi_{\alpha}(t) \leq \rho, \\
\rho, & \text{if } \Phi_{\alpha}(t) \geq \rho.
\end{cases}
\]

Then, for all \( \rho > 0 \),

\[
\int \int \Phi_{\alpha, \rho}(|x - y|) \, d\sigma(x) \, d\sigma(y) = \lim_{n \to \infty} \int \int \Phi_{\alpha, \rho}(|x - y|) \, d\sigma_n(x) \, d\sigma_n(y) \leq \liminf_{n \to \infty} I_\alpha(\sigma_n).
\]

The remark follows now from the monotone convergence theorem (as \( \rho \to \infty \)).
Lemma 6. ([L, p.79]) If $0 < \alpha < 1$ and $\sigma \in \mathcal{M}_\alpha$, then $I_\alpha(\sigma) \geq 0$. Moreover
\[ I_\alpha(\sigma) = 0 \quad \text{if and only if} \quad \sigma \equiv 0. \]

Lemma 6'. ([L, p.80]) If $\sigma \in \mathcal{M}_0$ and the support of $\sigma$ is contained in an open disk of radius 1, then $I_0(\sigma) \geq 0$ and
\[ I_0(\sigma) = 0 \quad \text{if and only if} \quad \sigma \equiv 0. \]

These two lemmas allow us to define an inner product in $\mathcal{M}_\alpha$ (for $0 < \alpha < 1$) and e.g. in $\mathcal{M}_0(\{|z| = 1/2\})$ (for $\alpha = 0$) as follows
\[ \langle \sigma, \gamma \rangle = \int \int \Phi_\alpha(|x-y|) \, d\sigma(x) d\gamma(y). \]

Observe that the associated norm verifies
\[ \|\sigma\|^2 = I_\alpha(\sigma). \]

Lemma 7. If $0 < \alpha < 1$, $\{\sigma_n\} \subset \mathcal{M}_\alpha^+(\partial\Delta)$, $\sigma_n \overset{w^*}{\rightarrow} \sigma$, and $\|\sigma_n\| < M$, then $\sigma \in \mathcal{M}_\alpha^+(\partial\Delta)$ and $\sigma_n \overset{w}{\rightarrow} \sigma$ in weak sense with respect to the inner product.

The result remains true if we replace $\mathcal{M}_\alpha^+(\partial\Delta)$ by $\mathcal{M}_0^+(\{|z| = 1/2\})$.

Proof. From Remark 1, we get that $\sigma \in \mathcal{M}_\alpha^+(\partial\Delta)$ in the case $0 < \alpha < 1$ and $\sigma \in \mathcal{M}_0^+(\{|z| = 1/2\})$ in the case $\alpha = 0$. Moreover,
\[ \|\sigma\| \leq \liminf_{n \to \infty} \|\sigma_n\|. \]

Now, if $\gamma$ is a signed measure whose potential
\[ U_\alpha^\gamma(x) = \int \Phi_\alpha(|x-y|) \, d\gamma(y) \]
is a continuous function, then
\[ \langle \sigma_n, \gamma \rangle = \int U_\alpha^\gamma(x) \, d\sigma_n(x) \overset{n \to \infty}{\longrightarrow} \int U_\alpha^\gamma(x) \, d\sigma(x) = \langle \sigma, \gamma \rangle. \]

This demonstrates the lemma, since such measures are dense in $\mathcal{M}_\alpha(\partial\Delta)$ ($0 < \alpha < 1$) ([L, p.82]) and in $\mathcal{M}_0(\{|z| = 1/2\})$ ([L, p.83]). \qed
Lemma 8. If \(0 < \alpha < 1\), \(\{\sigma_n\}\) is a Cauchy sequence (with respect to the inner product) in \(\mathcal{M}_n^+(\partial \Delta)\) and \(\sigma_n \rightharpoonup \sigma\), then \(\sigma_n \rightarrow \sigma\) in the norm, i.e.,
\[
\|\sigma_n - \sigma\| \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

The statement remains true if we replace \(\mathcal{M}_n^+(\partial \Delta)\) by \(\mathcal{M}_n^+(\{|z| = 1/2\})\).

Proof. Since \(\{\sigma_n\}\) is a Cauchy sequence, \(\{\|\sigma_n\|\}\) is a bounded set and hence, and so, from Lemma 7, \(\sigma \in \mathcal{M}_n^+(\partial \Delta)\) and
\[
\|\sigma - \sigma_n\|^2 = \langle \sigma - \sigma_n, \sigma - \sigma_n \rangle = \lim_{p \rightarrow \infty} \langle \sigma - \sigma_n, \sigma_p - \sigma_n \rangle.
\]

Therefore, by Cauchy-Schwarz inequality,
\[
\|\sigma - \sigma_n\|^2 \leq \lim_{p \rightarrow \infty} \|\sigma - \sigma_n\| \|\sigma_p - \sigma_n\|,
\]
i.e.,
\[
\|\sigma - \sigma_n\| \leq \lim_{p \rightarrow \infty} \|\sigma_p - \sigma_n\|.
\]

Lemma 9. If \(0 < \alpha < 1\), \(K\) is a compact subset of \(\partial \Delta\) and \(\mu_K\) is the equilibrium measure of \(K\), then for any probability \(\sigma\) supported on \(K\),
\[
\|\sigma - \mu_K\|^2 \leq \|\sigma\|^2 - \|\mu_K\|^2.
\]

The lemma remains true for the case \(\alpha = 0\) if \(K\) is a compact subset of \(\{|z| = 1/2\}\).

Proof. Consider for \(t \in [0, 1]\) the probabilities supported on \(K\) given by
\[
\sigma_t = (1 - t)\mu_K + t\sigma.
\]

As \(\sigma_t = \mu_K + t(\sigma - \mu_K)\), we have that
\[
\|\mu_K\|^2 \leq \|\sigma_t\|^2 = \|\mu_K\|^2 + 2t \langle \mu_K, \sigma - \mu_K \rangle + t^2 \|\sigma - \mu_K\|^2.
\]

Hence, \(2 \langle \mu_K, \sigma - \mu_K \rangle + t\|\sigma - \mu_K\|^2 \geq 0\) if \(t \in (0, 1]\). By making \(t \rightarrow 0\) we obtain that \(\langle \mu_K, \sigma - \mu_K \rangle \geq 0\), i.e.,
\[
\langle \mu_K, \sigma \rangle \geq \|\mu_K\|^2.
\]

Hence,
\[
\|\sigma - \mu_K\|^2 = \|\sigma\|^2 - 2\langle \sigma, \mu_K \rangle + \|\mu_K\|^2
\]
\[
\leq \|\sigma\|^2 - 2\|\mu_K\|^2 + \|\mu_K\|^2 = \|\sigma\|^2 - \|\mu_K\|^2.
\]

\(\square\)
Now we can prove the characterization of $\text{cap}_\alpha(E)$ in terms of the $\alpha$-energy.

**Lemma 10.** If $0 < \alpha < 1$ and $E$ is any bounded Borel set in $\mathbb{C}$,

$$(6) \quad \frac{1}{\text{cap}_\alpha(E)} = \inf \left\{ I_\alpha(\mu) : \mu \text{ probability, } \mu(E) = 1 \right\},$$

and there exists a probability $\mu_e$ supported on $\overline{E}$ such that

$$(7) \quad \frac{1}{\text{cap}_\alpha(E)} = I_\alpha(\mu_e).$$

**Proof.** If $\text{cap}_\alpha(E) = 0$ it is easy to check that both sides of (6) are simultaneously infinite. Assume $\text{cap}_\alpha(E) > 0$ and let $\mu$ be a probability with $\mu(E) = 1$ and finite $\alpha$-energy. Let $K_n \subset E$ be a sequence of compact sets such that $\mu(K_n) \to 1$. Then

$$\frac{1}{\text{cap}_\alpha(E)} \leq \frac{1}{\text{cap}_\alpha(K_n)} \leq I_\alpha(\mu|_{K_n}) = \frac{1}{(\mu(K_n))^2} \int \int_{K_n \times K_n} \Phi_\alpha(|x-y|) \, d\mu(x) \, d\mu(y).$$

Letting $n \to \infty$, the monotone convergence theorem gives us

$$\frac{1}{\text{cap}_\alpha(E)} \leq I_\alpha(\mu)$$

and so

$$\frac{1}{\text{cap}_\alpha(E)} \leq \inf \left\{ I_\alpha(\mu) : \mu \text{ probability, } \mu(E) = 1 \right\}.$$

In order to prove the opposite inequality we choose an increasing sequence $K_n \subset E$ of compact sets such that $\text{cap}_\alpha(K_n) \to \text{cap}_\alpha(E)$. Then, if $\mu_n$ denotes the equilibrium distribution of $K_n$, $I_\alpha(\mu_n) = \frac{1}{\text{cap}_\alpha(K_n)} \leq \frac{1}{\text{cap}_\alpha(E)}$. This finishes the proof of (6). Moreover, up on extracting a subsequence, we can assume that $\mu_n \overset{w^*}{\to} \mu_e$, where $\mu_e$ is a probability measure supported on $\overline{E}$. By Remark 1 we see that this measure satisfies

$$(8) \quad I_\alpha(\mu_e) \leq \lim_{n \to \infty} I_\alpha(\mu_n) = \frac{1}{\text{cap}_\alpha(E)}.$$

To finish we only need to check that $\mu_e$ is the measure appearing in (7), by proving that, in fact, we have equality in (8). Since

$$\left| I_\alpha(\mu_e) - I_\alpha(\mu_n) \right| = \left| \|\mu_e\|^2 - \|\mu_n\|^2 \right|$$
and

$$\|\mu_c\| = \|\mu_n\| = \|\mu_c - \mu_n\|,$$

in view of Lemma 8, we only need to show that \(\{\mu_n\}\) is a Cauchy sequence in \(\mathcal{M}^+_0(E)\). But if \(n, p \in \mathbb{N}\) and \(n \geq p\), as \(K_p \subset K_n\), we have that \(\mu_p\) is a probability supported on \(K_n\), and so by Lemma 9 and (8),

$$\|\mu_p - \mu_n\|^2 \leq \|\mu_p\|^2 - \|\mu_n\|^2 = I_{\alpha}(\mu_p) - I_{\alpha}(\mu_n) \xrightarrow{p, n \to \infty} 0. \quad \square$$

For the case \(\alpha = 0\) we have

**Lemma 10’**. If \(E\) is a Borel subset of \(\partial\Delta\). Then

$$\log \frac{1}{\text{cap}_0(E)} = \inf \{I_0(\mu) : \mu \text{ probability, } \mu(E) = 1\}.$$  \quad (9)

Moreover, there exists a probability measure \(\mu_c\) supported on \(\overline{E}\) such that

$$\log \frac{1}{\text{cap}_0(E)} = I_0(\mu_c).$$  \quad (10)

**Proof.** The proof follows the same lines as in Lemma 10, up to the final steps. In order to prove (10) we need some modifications.

For \(\lambda > 0\), and \(A \subset \mathbb{C}\), we will denote by \(\lambda A\) the set \(\lambda A = \{\lambda z : z \in A\}\).

If \(E\) is a Borel subset of \(\partial\Delta\), then \(\frac{1}{2}E\) is a Borel subset of \(\{|z| = 1/2\}\). Now, take an increasing sequence of compact subsets \(K_n^* \subset \frac{1}{2}E\) such that \(\text{cap}_\alpha(K_n^*) \to \text{cap}_\alpha(\frac{1}{2}E)\). Let \(\mu_c^*\) be the equilibrium measure of \(K_n^*\). The probabilities \(\mu_n\) defined by

$$\mu_n(A) = \mu_c^*\left(\frac{1}{2}A\right), \quad \text{for } A \text{ Borel subset of } \partial\Delta,$$

are supported on \(E\). Moreover it is clear that

$$I_0(\mu_c^*) = I_0(\mu_n) + \log 2. \quad (12)$$

On the other hand as in the proof of Lemma 10, \(\mu_c^* \rightharpoonup \mu_c^*\), where \(\mu_c^*\) is a probability measure supported on \(\frac{1}{2}E\), and

$$\lim_{n \to \infty} I_0(\mu_n^*) = I_0(\mu_c^*). \quad (13)$$

As \(\mu_n \rightharpoonup \mu_c\), where \(\mu_c\) is the probability supported on \(\overline{E}\) related with \(\mu_c^*\) by (11), we conclude from (12) and (13) that

$$\lim_{n \to \infty} I_0(\mu_n) = I_0(\mu_c).$$
This finishes the proof of Lemma 10'.

Now we are ready to finish the proof of Theorem 1. Let $E$ be a Borel subset of $\partial \Delta$ such that

\[
\text{cap}_\alpha(f^{-1}(E)) = \text{cap}_\alpha(E) > 0.
\]

We choose an increasing sequence of compact sets $K_n \subset E$ such that $\text{cap}_\alpha(K_n) \nearrow \text{cap}_\alpha(E)$. Let $\mu_n$ be the $\alpha$-equilibrium measure of $K_n$ and let $\mu_e$ be the probability measure supported on $\overline{E}$ given by Lemmas 10 and 10'. We have (from the proof of these lemmas) that

\[
\mu_n \overset{w^*}{\to} \mu_e \quad \text{and} \quad I_\alpha(\mu_n) \searrow I_\alpha(\mu_e).
\]

as $n \to \infty$. In fact,

\[
\|\mu_n - \mu_e\| \to 0, \quad \text{as} \quad n \to \infty.
\]

Let $\nu_n$ be the probability measure, with $\nu_n(f^{-1}(K_n)) = 1$, such that $P_{\nu_n} = P_{\mu_n} \circ f$ (see Lemma 3). We can suppose up to extracting a subsequence, that $\nu_n \overset{w^*}{\to} \nu$ where $\nu$ is a probability on $\overline{f^{-1}(E)}$. As the Poisson kernel is continuous in $\Delta$ we obtain, by using the $w^*$-convergence, that

\[
P_{\mu_n} \to P_{\mu_e} \quad \text{and} \quad P_{\nu_n} \to P_{\nu}, \quad \text{as} \quad n \to \infty,
\]

pointwisely. Therefore $P_{\nu} = P_{\mu_e} \circ f$ (and so $\nu$ is a probability on $f^{-1}(\overline{E})$).

**Claim**: $I_\alpha(\nu_n) \to I_\alpha(\nu)$ as $n \to \infty$.

As $\nu_n$ is a probability measure on $f^{-1}(E)$, Lemmas 10 and 10' guarantee that

\[
\frac{1}{\text{cap}_\alpha(f^{-1}(E))} \leq I_\alpha(\nu_n),
\]

and so, by taking $n \to \infty$, and since moreover $P_{\nu} = P_{\mu_e} \circ f$ (by Lemma 5)

\[
\frac{1}{\text{cap}_\alpha(f^{-1}(E))} \leq I_\alpha(\nu) \leq I_\alpha(\mu_e) = \frac{1}{\text{cap}_\alpha(E)}.
\]

From (14), we deduce that $I_\alpha(\nu) = I_\alpha(\mu_e)$. Finally, we can reason as in the case when $E$ is closed and we conclude that either $f$ is a rotation or $\mu_e$ is normalized Lebesgue measure, i.e., $\text{cap}_\alpha(E) = \text{cap}_\alpha(\partial \Delta)$.

**Proof of the Claim**: Consider first the case $0 < \alpha < 1$. Since $P_{\nu_n} - \nu_n = P_{\mu_e} - \mu_n \circ f$, by Lemma 5 we obtain that

\[
\|\nu_p - \nu_n\|^2 = I_\alpha(\nu_p - \nu_n) \leq I_\alpha(\mu_p - \mu_n) = \|\mu_p - \mu_n\|^2 \xrightarrow{p,n \to \infty} 0.
\]
Therefore \( \{\nu_n\} \) is a Cauchy sequence in the norm and so, by Lemma 8, we have that
\[
\|\nu_n - \nu\| \to 0 \quad \text{and} \quad I_\alpha(\nu_n) \to I_\alpha(\nu)
\]
as \( n \to \infty \).

In the case \( \alpha = 0 \), let \( \mu_n^* \) and \( \nu_n^* \) be the measures defined from \( \mu_n \) and \( \nu_n \) as in the proof of Lemma 10'. Then using again Lemma 5 and (12) we have that
\[
\|\nu_n^* - \nu_n^*\|^2 = I_0(\nu_n^* - \nu_n^*) = I_0(\nu_p - \nu_n) + \log 2 \\
\leq I_0(\mu_p - \mu) + \log 2 = \|\mu_n^* - \mu_n^*\|^2 \xrightarrow{p,n \to \infty} 0.
\]

Therefore \( \{\nu_n^*\} \) is a Cauchy sequence in the norm and again by Lemma 8, we obtain that
\[
\|\nu_n^* - \nu^*\| \to 0 \quad \text{and} \quad I_0(\nu_n^*) \to I_0(\nu^*)
\]
as \( n \to \infty \). It follows, from (12) that
\[
I_0(\nu_n) \to I_0(\nu), \quad \text{as} \quad n \to \infty. \quad \square
\]

4. Some further results on distortion of capacity in the case \( n = 1 \).

First we show that Theorem 1 is sharp. In what follows \(|\cdot|\) will denote not normalized Lebesgue measure in \( \partial \Delta \) (\( |\partial \Delta| = 2\pi \)).

**Proposition 1.** \( \text{cap}_\alpha(f^{-1}(E)) \) can take any value between \( \text{cap}_\alpha(E) \) and \( \text{cap}_\alpha(\partial \Delta) \). More precisely, given \( 0 < s \leq t < \text{cap}_\alpha(\partial \Delta) \) there exist a Borel subset \( E \) of \( \partial \Delta \) and an inner function \( f \) with \( f(0) = 0 \) such that \( \text{cap}_\alpha(E) = s \) and \( \text{cap}_\alpha(f^{-1}(E)) = t \).

This is a corollary of the following lemma

**Lemma 11.** Let \( I \) be any closed interval in \( \partial \Delta \) with \( |I| > 0 \), and let \( B \) be a finite union of closed intervals in \( \partial \Delta \) such that \( |B| = |I| \). Then there exists an inner function \( f \) such that
\[
f(0) = 0, \quad \text{and} \quad f^{-1}(I) = B.
\]

In fact, if \( 0 < |I| < 2\pi \), then \( f \) is unique.

As a matter of fact, if \( B \) is any Borel set the proof of Lemma 11 gives also \( f \) as above with \( f^{-1}(I) = B \), where \( = \) denotes equality up to a set of zero Lebesgue measure.

It is natural to wonder if this lemma holds in higher dimensions, more precisely:
Is it true that given an interval $I$ in $\partial \Delta$ and a Borel subset $B$ of $\mathbb{S}_n$ such that

$$\frac{|B|}{\omega_{2n-1}} = \frac{|I|}{2\pi},$$

there is an inner function $f: \mathbb{B}_n \to \Delta$ such that $f^{-1}(I) = B$?

It is no possible to construct such $f$ by using the Ryll-Wojtaszczyk polynomials (see [R2]), since in that case the following stronger result would be true too: Given $E, I$ subsets of $\partial \Delta$ with $|E| = |I|$ and $N \in \mathbb{N}$, there exists an inner function $f: \Delta \to \Delta$ such that

$$E = f^{-1}(I), \quad \text{and} \quad f^{(j)}(0) = 0, \quad \text{if } j \leq N.$$

But it is easy to see, as a consequence of Lemma 12, that this is no possible in general.

The following is well known.

**Corollary.** Let $0 \leq \alpha < 1$. If $I$ is any interval in $\partial \Delta$, then $I$ has the minimum $\alpha$-capacity between all the Borel subsets of $\partial \Delta$ with the same Lebesgue measure than $I$.

**Proof of Lemma 11.** Let $u$ be the Poisson integral of the characteristic function of $B$, and let $\tilde{u}$ be its conjugate harmonic function chosen such that $\tilde{u}(0) = 0$. Since $u(0) = |B|/2\pi$ the holomorphic function $F = u + i\tilde{u}$ transforms $\Delta$ into the strip $S = \{\omega : 0 < \text{Re} \omega < 1\}$. Notice that $F$ has radial boundary values except for a finite number of points, and $F$ applies the interior of $B$ into $\{\omega : \text{Re} \omega = 1\}$ and $\partial \Delta \setminus B$ into $\{\omega : \text{Re} \omega = 0\}$.

Now, let $G$ be the Riemann mapping of $S$ chosen such that

$$G(|B|/2\pi) = 0.$$  

$G$ transforms $\{\omega : \text{Re} \omega = 1\}$ onto an interval $J$ of $\partial \Delta$. On the other hand, the function $h = G \circ F$ is clearly an inner function, $h(0) = 0$ and $h^{-1}(I) = B$. By composing $h$ with an appropriate rotation we finish the proof of the existence statement.

To show the uniqueness of $f$, it is sufficient to prove the following

**Lemma 12.** If $A$ is any Borel subset of $\partial \Delta$, such that $\int_A e^{-i\theta} d\theta \neq 0$, and $f, g$ are inner functions with $f(0) = g(0) = 0$ such that

$$f^{-1}(A) \circ g^{-1}(A),$$

then $f \equiv g$. 
Here $\circ$ denotes equality up to a set of zero Lebesgue measure.

**Proof.** Let $F: \Delta \rightarrow \{ \omega : 0 < \Re \omega < 1 \}$ be the holomorphic function given by

$$F(z) = \frac{1}{2\pi} \int_A \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta.$$  

$F$ is univalent in a neighbourhood of 0, because

$$F'(0) = \frac{1}{\pi} \int_A e^{-i\theta} \, d\theta \neq 0.$$  

Now, observe that $\Re (F \circ f) = \Re (F \circ g)$ almost everywhere on $\partial \Delta$. Since $\Re (F \circ f)$ and $\Re (F \circ g)$ are bounded harmonic functions it follows that $F \circ f = F \circ g + ic$ in $\Delta$, where $c$ is a real constant. Since $f(0) = g(0)$, we deduce that $F \circ f = F \circ g$ which proves the lemma because $F$ is univalent in a neighbourhood of 0. □

Observe that, in particular, the condition $\int_A e^{-i\theta} \, d\theta \neq 0$ is satisfied e.g. if $A$ is any interval in $\partial \Delta$ with $0 < \vert A \vert < 2\pi$.

The condition $\int_A e^{-i\theta} \, d\theta \neq 0$ is not only a technicality. If $A$ is $k$-symmetrical (i.e., there exists a subset $A_0 \subset A$, with $A_0 \subset [0, 2\pi/k]$, such that $A = A_0 \cup (A_0 + 2\pi/k) \cup (A_0 + 4\pi/k) \cup \cdots \cup (A_0 + 2\pi(k-1)/k)$, and $\int_A e^{-ik\theta} \, d\theta \neq 0$, then $f = \omega g$, where $\omega$ is a $k$-th root of unity. To see this, one can use Lemma 12 with the functions $h \circ f$, $h \circ g$ and the set $h(A)$, where $h(z) = z^k$.

Also, note that if $A$ is the union of two intervals in $\partial \Delta$, then $f = \pm g$, because $\int_A e^{-i\theta} \, d\theta = 0$ implies that $A$ is 2-symmetrical.

Notice that if the function $g$ in Lemma 12 were the identity, and $0 < \vert A \vert < 2\pi$, then, by ergodicity, we have that $f$ is a rotation of rational angle. This, together the above remark, could suggest that perhaps the following statement is true:

*If $A$ is any Borel subset of $\partial \Delta$, such that $0 < \vert A \vert < 2\pi$, and $f$, $g$ are inner functions with $f(0) = g(0) = 0$ such that

$$f^{-1}(A) = g^{-1}(A),$$

then $f \equiv \lambda g$ with $\vert \lambda \vert = 1$.***

But this is false as the next example shows: Let $B$ be the following Blaschke product

$$B(z) = \frac{2z - 1}{2 - z}.$$

By applying a theorem of Stephenson [S, Theorem 3] to the pair $B, -B$, one obtains two inner functions $f$ and $g$ with $f(0) = g(0) = 0$, such that

$$B \circ f = -B \circ g.$$
But then \((B(f))^2 = (B(g))^2\), and so, if we had \(f = \lambda g\), we could conclude that 
\(B(z) = -B(\lambda z)\). But, since \(B'(0) \neq 0\), we had \(\lambda = -1\), i.e., \(B(z) = -B(-z)\), a contradiction.

The following is not unexpected since ergodic theory says that \(f^{-k}(E)\) is well
spread on \(\partial \Delta\). Hereafter \(f^k = f \circ \cdots \circ f\) denotes the \(k\)-iterate of \(f\) and \(f^{-k} = (f^k)^{-1}\).

**Proposition 2.** If \(f : \Delta \longrightarrow \Delta\) is inner but not a rotation, \(f(0) = 0\), \(0 \leq \alpha < 1\) and \(E\) is a Borel subset of \(\partial \Delta\) with \(\text{cap}_\alpha(E) > 0\), then 
\[
\text{cap}_\alpha(f^{-k}(E)) \rightarrow \text{cap}_\alpha(\partial \Delta) \quad \text{as} \quad k \to \infty.
\]

The proof of this result is an easy consequence of the following lemma.

**Lemma 13.** With the hypotheses of Proposition 2, if \(\mu\) is any probability on \(E\) with finite \(\alpha\)-energy and if \(\nu_k\) is the probability measure in \(f^{-k}(E)\) such that \(P_{\nu_k} = P_{\mu} \circ f^k\), then 
\[
I_\alpha(\nu_k) \longrightarrow I_\alpha\left(\frac{1}{2\pi}\right) \quad \text{as} \quad k \to \infty.
\]

With this, we have
\[
\frac{1}{\text{cap}_\alpha(f^{-k}(E))} \leq I_\alpha(\nu_k) \longrightarrow I_\alpha\left(\frac{1}{2\pi}\right) = \frac{1}{\text{cap}_\alpha(\partial \Delta)},
\]
and so the proof of Proposition 2 will be finished.

**Proof of Lemma 13.** We will prove it for \(0 < \alpha < 1\); the case \(\alpha = 0\) being similar.

By Lemma 2 (i), we have with an appropriate function \(g_\alpha\) that 
\[
I_\alpha(\sigma) = \int_0^1 \int_0^{2\pi} \left| P_\sigma(re^{i\theta}) \right|^2 d\theta g_\alpha(r) dr
\]
for any probability measure \(\sigma\) on \(\partial \Delta\).

Using (3) we have for all \(r \in (0,1)\) that 
\[
\int_0^{2\pi} \left| P_{\nu_k}(re^{i\theta}) \right|^2 d\theta \leq \int_0^{2\pi} \left| P_\mu(re^{i\theta}) \right|^2 d\theta.
\]

Since \(\mu\) has finite \(\alpha\)-energy, the right hand side in the last inequality, as a function, belongs to \(L^1(g_\alpha(r) dr)\). Therefore, by using the Lebesgue’s dominated convergence theorem, we would be done if we show that 
\[
(15) \quad \int_0^{2\pi} \left| P_{\nu_k}(re^{i\theta}) \right|^2 d\theta \longrightarrow \frac{1}{2\pi} \quad \text{as} \quad k \to \infty,
\]
for each $r$ with $0 < r < 1$.

By definition of $\nu_k$,

$$
\left| P_{\nu_k}(re^{i\theta}) - \frac{1}{2\pi} \right| \leq \frac{1}{2\pi} \int_{\partial\Delta} \left| \frac{1}{f_k(re^{i\theta}) - \xi} \right|^2 d\mu(\xi) - 1 \left| \frac{1}{2\pi} \right| d\mu(\xi).
$$

But, by Schwarz’s lemma, and since $f$ is not a rotation, $|f_k(re^{i\theta})| \to 0$ as $k \to \infty$, uniformly on $\theta$ for $r$ fixed.

Therefore, for each $r$, $P_{\nu_k}(re^{i\theta}) \to 1/2\pi$, as $k \to \infty$, uniformly on $\theta$, and this implies (15). \hfill \square

Even in the case when $\text{cap}_\alpha(E) = 0$, the sets $f^{-k}(E)$ are well spread on $\partial\Delta$.

**Proposition 3.** If $f : \Delta \to \Delta$ is an inner function (but not a rotation) with $f(0) = 0$, $E$ is any non-empty Borel subset of $\partial\Delta$, and $\mu$ is any probability measure on $E$, then for some absolute constant $C$ and a positive constant $A$ that only depends on $|f'(0)|$, we have that

$$
\left| \nu_k(I) - \frac{|I|}{2\pi} \right| < C e^{-Ak},
$$

for all interval $I \subset \partial\Delta$. In particular,

$$
\nu_k \to \frac{|I|}{2\pi}
$$

in the weak-* topology.

Here $\nu_k$ is the probability measure concentrated at $f^{-k}(E)$ such that $P_{\nu_k} = P_{\mu} \circ f^k$.

**Proof.** It is similar to the proof of Lemma 3 in [P], but using the fact that $P_{\nu_k} = P_{\mu} \circ f^k$ instead of Lemma 1 in [P]. \hfill \square

**Corollary.** If $f : \Delta \to \Delta$ is inner, then $f$ assume all the values in $\partial\Delta$.

**Remark.** This last result is true in any dimension as we can prove as a consequence of Lemma 15 (see Section 5).

Indeed, let $f : \mathbb{B}_n \to \Delta$ be an inner function. It is enough to prove that $f^{-1}\{1\} \neq \emptyset$. But,

$$
(16) \qquad u := \text{Re} \left( \frac{1 + f}{1 - f} \right) = \frac{1 - |f|^2}{1 - |f|^2} > 0, \quad \text{in } \mathbb{B}_n.
$$
Therefore, \( u \) is harmonic and positive in \( \mathbb{B}_n \) and so there exists a positive measure in \( S_n \) such that
\[
\text{Re} \left( \frac{1 + f}{1 - f} \right) = P_\mu.
\]

By (16) \( P_\mu \) tends radially to 0 a.e. with respect to Lebesgue measure, since \( f \) is inner and (by Privalov’s theorem, see e.g., [R, Theorem 5.5.9]) \( f \) can assume the value 1 at most in a set of zero Lebesgue measure. Then, the Radon-Nikodym derivative of \( \mu \) with respect to Lebesgue measure is zero a.e., and so \( \mu \) is a singular measure.

By Lemma 15 it follows that \( P_\mu \to \infty \) in a set of full \( \mu \)-measure. But this is the same to say that \( f(r e^{i\theta}) \to 1 \) in that set.

When the inner function \( f \) has order \( k \geq 1 \) at 0, we can improve Theorem 1 in the case \( \alpha = 0 \).

**Theorem 3.** If \( f : \Delta \to \Delta \) is inner,
\[
f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0, \quad f^{(k)}(0) \neq 0, \quad (k \geq 1),
\]
and \( E \) is a Borel subset of \( \partial \Delta \), then
\[
(17) \quad \text{cap}_0(f^{-1}(E)) \geq \left( \text{cap}_0(E) \right)^{1/k}.
\]
Moreover, if \( \text{cap}_0(E) > 0 \), equality holds if and only if either \( f(z) = \lambda z^k \), with \( |\lambda| = 1 \), or \( \text{cap}_0(E) = \text{cap}_0(\partial \Delta) \).

**Proof.** For such a function \( f \), Schwarz’s lemma says us that \( |f(z)| \leq |z|^k \), with equality only if \( f(z) = \lambda z^k \) with \( |\lambda| = 1 \). With this in mind, subordination principle says now that if \( v \) is a subharmonic function in \( \Delta \), then
\[
\int_0^{2\pi} v(f(r e^{i\theta})) \, d\theta \leq \int_0^{2\pi} v(r^k e^{i\theta}) \, d\theta,
\]
with equality only if \( v \) is harmonic or \( f \) is a rotation of \( z^k \).

Now, in order to prove (17), we can assume that \( E \) is closed. If \( \mu_e \) is the equilibrium probability distribution of \( E \) and \( \nu \) is the probability in \( f^{-1}(E) \) such that \( P_\nu = P_\mu \circ f \), then
\[
I_0(\nu) = 2\pi \left( \int_0^1 \int_0^{2\pi} \left| P_{\mu_e}(f(r e^{i\theta})) \right| - \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{r} \right) dr \leq 2\pi \left( \int_0^1 \int_0^{2\pi} \left| P_{\mu_e}(r^k e^{i\theta}) \right| - \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{r} \right) dr.
\]
Substituting \( r^k = t \), we obtain that
\[
I_0(\nu) \leq \frac{1}{k} I_0(\mu_e).
\]
This finishes the proof of (17). The equality statement can be proved in the same way that in Theorem 1.

\[ \frac{1}{\text{cap}_\alpha(f^{-1}(E))} - \frac{1}{\text{cap}_\alpha(\partial \Delta)} \leq \frac{C_\alpha}{k^{1-\alpha}} \left( \frac{1}{\text{cap}_\alpha(E)} - \frac{1}{\text{cap}_\alpha(\partial \Delta)} \right). \]

where \( C_\alpha \) is a constant depending only on \( \alpha \).

We expect \( C_\alpha = 1 \), but we have not been able to show this.

5. Distortion of \( \alpha \)-content.

The following is an extension of Löwner’s lemma.

**Theorem 4.** If \( f : \mathbb{R}^n \rightarrow \Delta \) is inner, \( f(0) = 0 \) and \( E \) is a Borel subset of \( \partial \Delta \), then, for \( 0 < \alpha \leq 1 \),

\[ M_{2n-2+\alpha}(f^{-1}(E)) \geq C_{\alpha} M_{\alpha}(E) \]

and

\[ M_{2(n-1+\alpha)}(f^{-1}(E)) \geq C'_{\alpha} M_{\alpha}(E). \]

Here \( M_\beta \) and \( \mathcal{M}_\beta \) denote, respectively, \( \beta \)-dimensional content with respect to the euclidean metric and the metric in \( \mathbb{S}^n \) given by

\[ d(a, b) = |1 - \langle a, b \rangle|^{1/2}, \]

where \( \langle a, b \rangle = \sum a_j \overline{b}_j \) is the inner product in \( \mathbb{C}^n \). This metric is equivalent to the Carnot-Caratheodory metric in the Heisenberg group model for \( \mathbb{S}^n \). We refer to [R] for details about this metric.

Recall that in a metric space \((X, d)\) the \( \alpha \)-content of a set \( E \subset X \) is defined as

\[ M_{\alpha}(E) = \inf \left\{ \sum r_i^{\alpha} : E \subset \bigcup_i B_d(x_i, r_i) \right\}. \]

Observe that, as a consequence of Theorem 4, one obtains

**Corollary.** If \( f : \mathbb{R}^n \rightarrow \Delta \) is inner and \( E \) is a Borel subset of \( \partial \Delta \), then

\[ \text{Dim}(f^{-1}(E)) \geq 2n - 2 + \text{Dim}(E) \]
and
\[ \text{Dim}(f^{-1}(E)) \geq 2n - 2 + 2 \text{Dim}(E) \]

where Dim and Dim denote, respectively, Hausdorff dimension with respect to the euclidean metric and the metric d.

In order to prove Theorem 4 we will prove a lemma about Poisson integrals. We need to consider the classical Poisson kernel (not normalized)
\[ P(\xi, z) = \frac{1 - |z|^2}{|\xi - z|^{2n}} \quad (z \in \mathbb{B}_n, \, \xi \in \mathbb{S}_n), \]

and the invariant Poisson kernel
\[ Q(\xi, z) = \frac{(1 - |z|^2)^n}{|1 - \langle z, \xi \rangle|^{2n}} \quad (z \in \mathbb{B}_n, \, \xi \in \mathbb{S}_n). \]

Of course, they coincide if \( n = 1 \). In this section if \( \nu \) is a positive measure in \( \mathbb{S}_n \), we will denote by \( P_\nu \)
\[ P_\nu(z) = \int_{\mathbb{S}_n} P(\xi, z) \, d\nu(\xi) \]
and by \( Q_\nu \) the invariant Poisson extension of \( \nu \)
\[ Q_\nu(z) = \int_{\mathbb{S}_n} Q(\xi, z) \, d\nu(\xi). \]

**Lemma 14.** Let \( \mu \) be a finite positive measure in \( \partial \Delta \), and let \( f : \mathbb{B}_n \to \Delta \) be an inner function. Then, there exists a finite measure \( \nu \geq 0 \) in \( \mathbb{S}_n \) such that \( P_\mu \circ f = P_\nu \), and if \( \nu \) has singular part \( \sigma \) and continuous part \( \gamma \), and we denote
\[ A = \{ \xi \in \mathbb{S}_n : P_\gamma(r\xi) \to \infty, \text{ as } r \to 1 \} \]
and
\[ B = \{ \xi \in \mathbb{S}_n : \exists \lim_{r \to 1} f(r\xi) = f(\xi), |f(\xi)| = 1 \text{ and } \lim_{r \to 1} P_\gamma(r\xi) > 0 \}, \]

then
\[ A \cup B \subset f^{-1}(\text{support } \mu) \]
and so
\[ \nu(f^{-1}(\text{support } \mu)) = \|\nu\|. \]

The same is true if we replace \( P_\nu \) by \( Q_\nu \) for (possibly) other measure \( \nu' \geq 0 \), and \( A, B \) by the following sets
\[ A' = \{ \xi \in \mathbb{S}_n : Q_{\nu'}(r\xi) \to \infty, \text{ as } r \to 1 \}, \]
and

\[ B' = \{ \xi \in \mathbb{S}_n : \exists \lim_{r \to 1} f(r\xi) = f(\xi), |f(\xi)| = 1 \text{ and } \lim_{r \to 1} Q_{\gamma'}(r\xi) > 0 \}, \]

where \( \sigma' \) and \( \gamma' \) denote, respectively, the singular and the continuous part of \( \nu' \).

**Proof.** We will prove the lemma only for the measure \( \nu' \), since the proof of the result for \( \nu \) is similar and more standard.

Let \( U : \Delta \to \mathbb{C} \) be a holomorphic function such that \( \text{Re} \ U = \mu \). Then \( U \circ f \) is also holomorphic and so \( \text{Re} \ (U \circ f) = \mu \circ f \) is pluriharmonic, i.e. harmonic and \( \mathcal{M} \)-harmonic (see e.g. [R, Theorem 4.4.9]). Therefore there exist finite positive measures \( \nu \) and \( \nu' \) in \( \mathbb{S}_n \) such that

\[ P_\mu \circ f = P_\nu, \quad P_\mu \circ f = Q_{\nu'}. \]

Let us denote by \( E \) the support of \( \mu \). If \( \xi \in A' \), then \( |f(r\xi)| \to 1 \) as \( r \to 1 \). The curve \( \{ f(r\xi) : \ 0 \leq r < 1 \} \) in \( \Delta \) must end on a unique point \( e^{i\psi} = f(\xi) \in \Delta \), since otherwise we would have \( P_\mu \equiv \infty \) on a set of positive Lebesgue measure. Now, \( e^{i\psi} \in E \), since otherwise \( P_\mu \) vanishes continuously at \( e^{i\psi} \). Therefore \( A' \subset f^{-1}(E) \).

Similarly one sees that \( B' \subset f^{-1}(E) \).

The set \( A' \) has full \( \sigma' \)-measure since by the inequality (20), that we will prove later,

\[ \{ \xi \in \mathbb{S}_n : \underline{D} \sigma'(\xi) = \infty \} \subset A', \]

where

\[ \underline{D} \sigma'(\xi) = \lim_{r \to 0} \inf \frac{\sigma'(B_d(\xi, r))}{|B_d(\xi, r)|}, \]

and the set \( \{ \xi : \underline{D} \sigma'(\xi) = \infty \} \) has full \( \sigma' \)-measure (see Lemma 15 below). Let us observe that ([R, p.67])

\[ |B_d(\xi, r)| \sim r^{2n}. \]

The set \( B' \) has full \( \gamma' \)-measure, since as \( r \to 1 \)

\[ Q_{\gamma'}(r\xi) \longrightarrow \frac{d\gamma'}{dL} \quad \text{a.e.} \]

with respect to Lebesgue measure \( L \) (see, e.g., [R, Theorem 5.4.9]) and \( \{ \frac{d\nu'}{dL} > 0 \} \) has full \( \gamma' \)-measure.

**Lemma 15.** Suppose that \( \mu \) is a singular positive Borel measure (with respect to Lebesgue measure) in \( \mathbb{S}_n \). Then

\[ \underline{D} \mu(x) = \infty \quad \text{a.e.} \ \mu. \]
Proof. Let \( A \) be a Borel set such that \( |A| = 0 \), and \( \mu \) is concentrated on \( A \). Define for \( \alpha > 0 \)

\[
A_\alpha = \{ x \in A : D \mu(x) < \alpha \}.
\]

It is enough to prove that \( \mu(A_\alpha) = 0 \), and by regularity that \( \mu(K) = 0 \) for all \( K \) compact subset of \( A_\alpha \).

Fix \( \varepsilon > 0 \). Since \( K \subset A_\alpha \subset A \), \( |K| = 0 \) and so there exists an open set \( V \) with \( K \subset V \) and \( |V| < \varepsilon \) (\(| \cdot | \) denotes Lebesgue measure).

Now, for each \( x \in K \), we can find \( r_x > 0 \) such that

\[
\frac{\mu(B_d(x, r_x))}{|B_d(x, r_x)|} < \alpha \quad \text{and} \quad B_d(x, r_x) \subset V.
\]

The family \( \{ B_d(x, r_x/3) : x \in K \} \) covers \( K \), hence we can extract a finite subcollection \( \Phi \) that also covers \( K \). Now, using a Vitali-type lemma (see, e.g., [R, Lemma 5.2.3]), we can find a disjoint subcollection \( \Gamma \) of \( \Phi \) such that

\[
K \subset \bigcup_{\gamma} B_d(x_\gamma, r_{x_\gamma}).
\]

Note that as a consequence of Proposition 5.1.4 in [R] we have that

\[
\Theta_d := \sup_{\delta} \frac{|B_d(x, r_x)|}{|B_d(x, r_x/3)|} < \infty.
\]

Therefore

\[
\mu(K) \leq \sum_{\gamma} \mu(B_d(x_\gamma, r_{x_\gamma})) < \alpha \sum_{\gamma} |B_d(x_\gamma, r_{x_\gamma})| < \Theta_d \alpha \sum_{\gamma} |B_d(x_\gamma, r_{x_\gamma}/3)| \leq \Theta_d \alpha |V| < \Theta_d \alpha \varepsilon \quad \Box.
\]

Proof of Theorem 4. We will prove only (ii), since (i) is obtained in a similar way.

Assume, as we may, that \( E \) is a closed subset of \( \partial \Delta \) and \( M_\alpha(E) > 0 \). Then, see e.g. [T, p.64], there exists a positive mass distribution on \( E \) of finite total mass, such that:

(a) \( \mu(E) = M_\alpha(E) \),

(b) \( \mu(I) \leq C_\alpha |I|^\alpha \) for any open interval \( I \), where \( C_\alpha \) is a constant independent of \( E \). An standard argument shows that

\[
P_\mu(z) \leq \frac{C_\alpha}{(1 - |z|)^{1-\alpha}}, \quad (z \in \Delta),
\]

with \( C_\alpha \) a new constant. Let \( \nu' \geq 0 \) be a measure in \( S_n \) such that \( P_\mu \circ f = Q_{\nu'} \). Schwarz’s lemma (see e.g. [R, Theorem 8.1.2]) and (18) give the same inequality for \( \nu \),

\[
Q_{\nu'}(z) \leq \frac{C_\alpha}{(1 - \|z\|)^{1-\alpha}}, \quad (z \in \mathbb{B}_n).
\]
We claim now that also we have

\begin{equation}
Q_\nu(z) \geq C_n \frac{\nu'(B_d(\xi, (2(1 - \|z\|))^{1/2}))}{(1 - \|z\|)^n}, \quad (z \in \mathbb{B}_n),
\end{equation}

where \( \xi = z/\|z\| \) and \( B_d(\xi, R) \) denotes the \( d \)-ball with center \( \xi \) and radius \( R \).

Now, using (19) and (20), we obtain that

\begin{equation}
\nu'(B_d(\xi, R)) \leq C_{n,\alpha} R^{2(n-1+\alpha)}, \quad (\xi \in \mathbb{S}_n, R > 0).
\end{equation}

If we cover the set \( A' \cup B' \) (see Lemma 14) with \( d \)-balls of radii \( R_i \), we see by (21) that

\[ \nu'(A' \cup B') \leq C_{n,\alpha} \sum_i R_i^{2(n-1+\alpha)} \]

and so

\[ \|\nu'\| = \nu'(A' \cup B') \leq C_{n,\alpha} \mathcal{M}_{2(n-1+\alpha)}(A' \cup B') \leq C_{n,\alpha} \mathcal{M}_{2(n-1+\alpha)}(f^{-1}(E)). \]

So, since \( f(0) = 0 \),

\[ \mathcal{M}_\alpha(E) = \|\mu\| = \|\nu'\| \leq C_{n,\alpha} \mathcal{M}_{2(n-1+\alpha)}(f^{-1}(E)). \]

Therefore, in order to finish the proof, it remains only to prove (20). Observe first that we can assume that \( \xi = e_1 = (1,0,\ldots,0) \) since \( d \) is invariant under the unitary transformations of \( \mathbb{S}_n \) for the inner product \( \langle \cdot, \cdot \rangle \). Now, if \( z = re_1 \), write \( \delta^2 = 2(1 - r) \). If \( \eta \in B_d(e_1, \delta) \), then

\[ \|1 - r\eta\| \leq \|1 - \eta\| + \|\eta\|(1 - r) \leq 3(1 - r). \]

Hence, if \( \eta \in B_d(e_1, \delta) \)

\[ Q(\eta, z) = \left( \frac{1 - r^2}{\|1 - r\eta\|^2} \right)^n \geq \frac{9^{-n}}{(1 - r)^n}. \]

Since \( Q \) is invariant under the action of the unitary group for the inner product \( \langle \cdot, \cdot \rangle \) in \( \mathbb{S}_n \), we obtain that if \( z = r\xi \) and \( \eta \in B_d(\xi, \delta) \), then

\[ Q(\eta, z) \geq \frac{9^{-n}}{(1 - r)^n}. \]

Finally,

\[ Q_\nu(z) \geq \int_{B_d(\xi, \delta)} Q(\eta, z) \, d\nu(\eta) \geq 9^{-n} \frac{\nu(B_d(\xi, \delta))}{(1 - r)^n}. \]

\[ \square \]
6. An open question.

We have discussed how inner functions distort boundary sets. There are some results on how they distort subsets of $\Delta$. On the one hand Hamilton [H] has shown that for all Borel subset $E$ of $\Delta$,

$$H_\alpha(f^{-1}(E)) \geq H_\alpha(E), \quad 0 < \alpha \leq 1,$$

where $H_\alpha$ denotes $\alpha$-Hausdorff measure.

One naturally expects the following to be true:

If $f : \Delta \to \Delta$ is inner, $f(0) = 0$ and $E$ is a Borel subset of $\Delta$, then

$$\text{cap}_\alpha(f^{-1}(E)) \geq \text{cap}_\alpha(E).$$

This we can prove only if $\alpha = 0$. The idea comes from [P1, p.336].

**Theorem 5.** Let $f : \Delta \to \Delta$ be an inner function. If for some $k \geq 1$

$$f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0, \quad f^{(k)}(0) \neq 0,$$

then,

$$\text{cap}_0(f^{-1}(E)) \geq (\text{cap}_0(E))^{1/k},$$

for all Borel subsets of $\Delta$. Moreover, this inequality is sharp.

**Sketch of proof.** By approximation, it is enough to prove it if $E$ is closed and $f$ is a finite Blaschke product. Let $f$ be

$$f(z) = z^k \prod_{j=1}^d e^{i\nu_j} \frac{z - a_j}{1 - \overline{a}_j z}.$$  

Denote by $g_E, g_F$ the Green’s functions of $\mathbf{C} \setminus E$ and $\mathbf{C} \setminus F$ (here $F = f^{-1}(E)$) with pole at $\infty$. Therefore,

$$g_E(z) - \log |z| = \log \frac{1}{\text{cap}_0(E)} + O(|z|^{-1}),$$

$$g_F(z) - \log |z| = \log \frac{1}{\text{cap}_0(F)} + O(|z|^{-1}),$$

as $|z| \to \infty$. Moreover, since $k \geq 1$

$$g_E(f(z)) - k \log |z| = \log \frac{1}{\text{cap}_0(E)} + O(|z|^{-1}),$$
as $|z| \to \infty$. It is easy to see that

$$g_{E}(f(z)) - \sum_{j=1}^{d} g_{F}(z, \overline{a}_{j}^{-1})$$

is harmonic in $\mathbb{C} \setminus (F \cup (\cup_{j=1}^{d} \{ \overline{a}_{j}^{-1} \}))$ and it is bounded at the points $\overline{a}_{j}^{-1}$ (here $g_{F}(z, \overline{a}_{j}^{-1})$ denotes the Green’s function of $\hat{\mathbb{C}} \setminus F$ with pole at $\overline{a}_{j}^{-1}$). Therefore, the function

$$(22) \quad G(z) = \frac{1}{k} g_{E}(f(z)) - g_{F}(z) - \frac{1}{k} \sum_{j=1}^{d} g_{F}(z, \overline{a}_{j}^{-1})$$

is harmonic in $\hat{\mathbb{C}} \setminus F$. Since $G = 0$ on $F$, it follows that $G \equiv 0$.

Now, by using the symmetry of Green’s function, we have that

$$g_{F}(z, \overline{a}_{j}^{-1}) \longrightarrow g_{F}(-\overline{a}_{j}^{-1}), \quad \text{as} \quad |z| \to \infty,$$

and so, from (22),

$$(23) \quad \log \frac{1}{\text{cap}_{0}(E)} - \log \prod_{j=1}^{d} |a_{j}| - k \log \frac{1}{\text{cap}_{0}(F)} - \sum_{j=1}^{d} g_{F}(\overline{a}_{j}^{-1}) = 0.$$

On the other hand, since $F \subset \Delta$, the maximum principle says that

$$g_{F}(z) \geq g_{\Delta}(z) = \log |z|.$$

Hence, from (23), we obtain that

$$\log \frac{1}{\text{cap}_{0}(E)} - \log \prod_{j=1}^{d} |a_{j}| - k \log \frac{1}{\text{cap}_{0}(F)} \geq \sum_{j=1}^{d} \log |a_{j}|^{-1},$$

and the inequality in the theorem follows. Finally, the inequality is sharp as is shown by taking $f(z) = z^{k}$, and $E$ an arc on $\{|z| = r\}$, $0 < r < 1$. □

References.


