Capacity distortion by inner functions
in the unit ball of $\mathbb{C}^n$

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1. Introduction.

An inner function is a bounded holomorphic function from the unit ball $B_n$ of $\mathbb{C}^n$ into the unit disk $\Delta$ of the complex plane such that the radial boundary values have modulus 1 almost everywhere. If $E$ is a non-empty Borel subset of $\partial \Delta$, we denote by $f^{-1}(E)$ the following subset of the unit sphere $S_n$ of $\mathbb{C}^n$

$$f^{-1}(E) = \{ \xi \in S_n : \lim_{r \to 1} f(r\xi) \text{ exists and belongs to } E \}. $$

There is a classical lemma of Löwner, see e.g. [R, p. 405], [T, p. 322], about the distortion of boundary sets under inner functions:

**Löwner’s lemma.** An inner function $f$, with $f(0) = 0$, is a measure preserving transformation when viewed as a mapping from $S_n$ to $\partial \Delta$, i.e. if $E$ is a Borel subset of $\partial \Delta$ then $|f^{-1}(E)| = |E|$, where in each case $| \cdot |$ means the corresponding normalized Lebesgue measure.

Here we extend this result to fractional dimensions as follows:

**Theorem 1.** If $f$ is inner in the unit ball of $\mathbb{C}^n$ ($n \geq 1$), $f(0) = 0$, and $E$ is a Borel subset of $\partial \Delta$, we have:

i) If $0 < \alpha < 2$, (and also $\alpha = 0$ if $n = 1$), then

$$\text{cap}_{2n-2+\alpha}(f^{-1}(E)) \geq C(n, \alpha) \text{cap}_\alpha(E).$$

ii) If $\alpha = 0$ and $n > 1$, then

$$\frac{1}{\text{cap}_{2n-2}(f^{-1}(E))} \leq C(n) \left(1 + \log \frac{1}{\text{cap}_0(E)} \right).$$

Here $\text{cap}_\alpha$ and $\text{cap}_0$ denote, respectively, $\alpha$-dimensional Riesz capacity and logarithmic capacity with respect to the distance in $S_n$ given by

$$d(a, b) = |1 - \langle a, b \rangle|^{1/2},$$

where

$$\langle a, b \rangle = \sum_{j=1}^{n} a_j \overline{b_j}$$

is the usual inner product in $\mathbb{C}^n$. This non-isotropic distance is the natural one in the analysis of problems concerning $S_n$. Also, this distance is equivalent to the Carnot-Carathéodory distance in the Heisenberg group model for $S_n$. We refer to [R] for details about this distance. Also we refer to [C], [KS] and [L] for definitions and basic background on capacity.

Observe that, as a consequence of Theorem 1, one obtains

**Corollary.** If $f : B_n \to \Delta$ is inner and $E$ is a Borel subset of $\partial \Delta$, then

$$\text{Dim}(f^{-1}(E)) \geq 2n - 2 + \text{Dim}(E)$$

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where $\text{Dim}$ denotes \textit{Hausdorff dimension} with respect to the distance $d$.

Analogous results with the euclidean distance instead of $d$ were obtained in [FPR]. Also, for some applications of these results we refer to [FP1], [FP2] and [FPR].

The basic tool that we will use to prove (1.1) is a formula relating the $\alpha$-energy $J_\alpha$ of a complex measure $\mu$ (see [L] for basic background on this subject) with its invariant Poisson extension $\mathcal{P}_\mu$. This approach is due to Beurling [B].

**Theorem 2.** If $\mu$ is a complex measure supported on $S_n$, the unit sphere of $\mathbb{C}^n$, we have, for all $n \geq 1$ and $0 < \alpha < 2n$, that

\[ J_\alpha(\mu) \asymp \int_0^1 \left\{ \int_{S_n} |\mathcal{P}_\mu(r\xi)|^2 \, d \xi \right\} r^{\alpha/2-1}(1 - r^2)^{(n-\alpha)/2-1} \, dr. \]  

(1.4)

Here and hereafter the expression $A \asymp B$ will mean that the quotient $A/B$ is bounded above and below by constants which can depend at most on $n$ and $\alpha$.

Recall that the invariant Poisson extension $\mathcal{P}_\mu$ of a complex measure $\mu$ (supported in $S_n$) is defined as follows

\[ \mathcal{P}_\mu(z) = \int_{S_n} \mathcal{P}(z, w) \, d\mu(w), \quad z \in B_n, \]

where

\[ \mathcal{P}(z, w) = \frac{(1 - |z|^n)^n}{\omega_{2n} |1 - \langle z, w \rangle|^n}, \quad z \in B_n, \quad w \in S_n, \]

is the Poisson-Szegö kernel ([R, p. 40], [F]) and $\omega_{2n}$ is the area of $S_n$. Observe that if $\alpha = 1$, the Poisson-Szegö kernel is simply the classical Poisson kernel.

Also, if $\mu$ is a complex measure on $S_n$, and $0 \leq \alpha < 2n$, then the $\alpha$-energy $J_\alpha(\mu)$ of $\mu$ is defined as

\[ J_\alpha(\mu) = \iint_{S_n \times S_n} \Phi_\alpha(d(\xi, \eta)) \, d\mu(\xi) \, d\mu(\eta), \]

where

\[ \Phi_\alpha(t) = \begin{cases} 
\log \frac{1}{t}, & \text{if } \alpha = 0, \\
\frac{1}{t^{\alpha}}, & \text{if } 0 < \alpha < 2n. 
\end{cases} \]

If $E$ is a closed subset of $S_n$, then

\[ (\text{cap}_\alpha(E))^{-1} = \inf \{ J_\alpha(\mu) : \mu \text{ a probability measure supported on } E \}, \]

for $0 < \alpha < 2n$,

\[ \log \frac{1}{\text{cap}_0(E)} = \inf \{ J_0(\mu) : \mu \text{ a probability measure supported on } E \}, \]

and the infimum is attained by a unique probability measure $\mu_\alpha$ which is called the \textit{equilibrium distribution} of $E$. 

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If \( E \) is any Borel subset of \( S_n \), then the \( \alpha \)-capacity of \( E \) is defined as

\[
\text{cap}_\alpha (E) = \sup \{ \text{cap}_\alpha (K) : K \subset E, \text{ } K \text{ compact} \}.
\]

The analogue of (1.4) with the euclidean distance instead of \( d \) was obtained in [FPR]; it is remarkable that in [FPR] equality is obtained with an explicit constant (see Theorem B below).

In order to prove Theorem 2 we will need a result about the integral of the square of a hypergeometric function that appears in [PR].

**Theorem A.** [PR] For all non negative integers \( p, q, n (n \geq 1) \) and for all \( \beta = \alpha/4, (0 < \beta < n/2) \), we have

\[
\int_0^1 \left( \frac{F(t)}{F(1)} \right)^2 t^{p+q+\beta-1}(1-t)^{n-2\beta-1} \, dt \leq \left[ \frac{\Gamma(p+\beta)\Gamma(q+\beta)}{\Gamma(p+n-\beta)\Gamma(q+n-\beta)} \right],
\]

where \( F(t) = F(p, q; p + q + n; t) \).

By \( F(a, b; c; t) \) we denote the usual Gauss hypergeometric function

\[
F(a, b; c; t) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k} \frac{t^k}{k!},
\]

where \((u)_k\) is the Pochhammer symbol,

\[
(u)_k = u(u+1)\cdots(u+k-1) = \frac{\Gamma(u+k)}{\Gamma(u)},
\]

and \(\Gamma(\cdot)\) denotes the Gamma function.

The outline of this paper is as follows: In Section 2 we will prove Theorem 2. Theorem 1 will be proved in Section 3.

We would like to thank J. L. Fernández for many useful discussions.

**Notations.** By \( C \) we will denote a constant, depending at most on \( n \) and \( \alpha \), which can change its value from line to line and even in the same line.

**2. Proof of Theorem 2.**

If we use the kernel \( \Phi_\alpha(\|K - \eta\|) \) instead of \( \Phi_\alpha(d(\xi, \eta)) \) we obtain the classical \( \alpha \)-dimensional Riesz energy that we will denote by \( I_\alpha(\mu) \). Observe that if \( n = 1 \), then \( I_{\alpha/2}(\mu) = J_\alpha(\mu) \), for all \( 0 < \alpha < 2 \), and also, \( I_0(\mu)/2 = J_0(\mu) \). This remark and the following theorem give the case \( n = 1 \) of Theorem 2, with equality for an appropriate constant instead of the symbol \( \asymp \).

**Theorem B.** [FPR] If \( \mu \) is a complex measure supported on \( \Sigma_{N-1} \), the unit sphere of \( \mathbb{R}^N \), and \( P_\mu \) is its classical Poisson extension (\( P_\mu = P_{\mu^c} \) if \( N = 2 \)), we have:
i) If \(0 < \alpha < N - 1\), then

\[
I_\alpha(\mu) = K(N, \alpha) \int_0^1 \left\{ \int_{\Sigma_{N-1}} |P_\mu(r\xi)|^2 \, d\xi \right\} r^{\alpha-1}(1 - r^2)^{N-2-\alpha} \, dr,
\]
with

\[
K(N, \alpha) = \frac{4\pi^{N/2}}{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{N-\alpha}{2}\right)}.
\]

ii) If \(m = \mu(\Sigma_{N-1})\), then

\[
I_0(\mu) = \omega_N \int_0^1 \int_{\Sigma_{N-1}} \frac{P_\mu(r\xi) - m}{\omega_N} \xi d\xi \left(1 - r^2\right)^{N-2} \frac{dr}{r} + \frac{4\pi}{2} \left[ \frac{\Gamma'}{\Gamma(\frac{N}{2})} - \frac{\Gamma'}{\Gamma(N - 1)} \right].
\]

where \(\omega_N\) denotes the area of \(\Sigma_{N-1}\). In particular, if \(N = 2\),

\[
I_0(\mu) = 2\pi \int_0^1 \int_0^{2\pi} \frac{P_\mu(re^{i\theta}) - \frac{m}{2\pi}}{dr} \, dl \, dr.
\]

Observe that in [FPR] Theorem B was proved only for signed measures, but the result, stated in the actual form, follows simply by splitting \(\mu\) into real and imaginary parts.

In [F], G. B. Folland obtained an expansion in spherical harmonics of the Poisson-Szegö kernel for the unit ball \(B_n\) in \(\mathbb{C}^n\). Let \(\Delta_{B_n}\) be the Laplace-Beltrami operator associated to the Bergman metric on \(B_n\).

\[
\Delta_{B_n} = \frac{4}{n+1} (1 - |z|^2) \sum_{i,j=1}^n (\delta_{ij} - \bar{z}_i z_j) \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.
\]

\(\Delta_{B_n}\) is the basic invariant differential operator on the symmetric space \(SU(n,1)/U(n) \approx B_n\). The solution of the Dirichlet problem

\[
\begin{cases}
\Delta_{B_n} u = 0, & \text{in } B_n, \\
u = f, & \text{in } \partial B_n,
\end{cases}
\]

with continuous boundary data \(f\), is given by the following representation formula

\[
u(z) = \int_{\partial B_n} P(z, w) f(w) \, dw.
\]

If \(\mathcal{H}^{p,q}\) denotes the linear space of restrictions to \(S_n\) of harmonic polynomials \(g(z, \xi)\) on \(\mathbb{C}^n\) which are homogeneous of degree \(p\) in \(z\) and degree \(q\) in \(\xi\), the solution of the Dirichlet problem (2.1) with \(f \in \mathcal{H}^{p,q}\) is given by

\[
u(r\xi) = S^{p,q}(r) f(\xi), \quad 0 \leq r \leq 1, \ \xi \in S_n,
\]

where

\[
S^{p,q}(r) = r^{p+q} \frac{F(p,q;p+q+n;r^2)}{F(p,q;p+q+n;1)}.
\]
The formula (2.2) gives to $S^{p,q}(r)$ a crucial role in order to obtain the expansion of the Poisson-Szegö kernel in spherical harmonics.

In [PR] we give uniform asymptotic estimates of these functions when $p, q$ grow to infinity:

**Theorem C** ([PR]). There exists a universal constant $C$, not depending on $n, q, m, z,$ such that, for all real numbers, $m, n, q \geq 1$, $q \geq \frac{1}{m}$, $0 \leq z < 1$, if we denote

$$G = F(mq, q; mq + q + n; z) B(mq, q + n),$$

where $B(x, y)$ is the usual Euler's Beta-function, then

$$G \geq C L$$

where

$$L = t_0^{m,q}(1 - m(1 - t_0))^q(1 - t_0)^n - 1 \left(\frac{1 - z}{a^2 - b^2 z}\right)^{1/4} \frac{1}{m \sqrt{q + 1}},$$

and

$$t_0 = \frac{a + b z - \sqrt{(1 - z)(a^2 - b^2 z)}}{2z} = \frac{2}{a + b z + \sqrt{(1 - z)(a^2 - b^2 z)}},$$

$$a = 1 + \frac{1}{m}, \quad b = 1 - \frac{1}{m}.$$ 

Besides, the inequality is sharp in the sense that

$$\lim_{q \to \infty} \frac{G}{L} = \sqrt{2\pi}.$$ 

Observe that without loss of generality we can suppose $m \geq 1$, because of the symmetry of the hypergeometric function in the two first parameters.

We summarize the results about these spherical harmonics (see e.g. [F]) in the following result. This Theorem generalizes the properties of classical spherical harmonics, which are described in [SW].

**Theorem D** [F]. For all $n \geq 2$, we have that:

i) $L^2(S_n)$ is the orthogonal sum $L^2(S_n) = \bigoplus_{p,q=0}^{\infty} \mathcal{H}^{p,q}$ and the dimension of $\mathcal{H}^{p,q}$ is

$$D = D(p, q; n) = \frac{(p + q + n - 1)(p + q + n - 2)!(q + n - 2)!}{p!q!(n - 1)!(n - 2)!}.$$ 

ii) If $f_1^{p,q}, f_2^{p,q}, \ldots, f_D^{p,q}$ is any orthonormal basis for $\mathcal{H}^{p,q}$, then

$$\sum_{j=1}^{D} f_j^{p,q}(\xi) \overline{f_j^{p,q}(\eta)} = \mathcal{H}^{p,q}(\langle \xi, \eta \rangle), \quad \xi, \eta \in S_n,$$

where $\mathcal{H}^{p,q}(\langle \cdot, \eta \rangle)$ is the zonal harmonic of degrees $p$ and $q$, and pole $\eta$. 

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iii) The $L^2$-norm of the function $H^{p,q}(\xi, \cdot)$ is

\begin{equation}
\int_{S_n} |H^{p,q}(\xi, \eta)|^2 \, d\eta = \frac{D}{\omega_{2n}}, \quad \text{for all } \xi \in S_n,
\end{equation}

where $d\eta$ denotes the usual Lebesgue surface measure in $S_n$ (not normalized).

iv) The function $H^{p,q}$ has the following explicit expression in terms of the Jacobi polynomials:

\begin{equation}
H^{p,q}(z) = \frac{D}{\omega_{2n}} \rho^{u-v} e^{i(p-q)\theta} \frac{P^{(n-2,u-v)}_v(n-2,u-v)}{P^{(n-2,u-v)}_v(1)}.
\end{equation}

where $z = \rho e^{i\theta}$, $u = \max\{p, q\}$, $v = \min\{p, q\}$ and $P^{(a,b)}_m(t)$ is the Jacobi polynomial of degree $m$ and parameters $a, b$ ([L1, p. 275], [AS, p. 785]; observe that in [F] there is a typing mistake in the definition of these polynomials).

Moreover, $\{H^{p,q}(z)\}_{p,q=0}^{\infty}$ is an orthogonal basis of $L^2(\{|z| < 1\})$ with respect to the measure $(1 - |z|^2)^{n/2} \, dz \, d\eta$ (since every polynomial in the variables $z$ and $\bar{z}$ can be expressed as a finite linear combination of $\{H^{p,q}(z)\}_{p,q=0}^{\infty}$).

v) For $0 \leq r < 1$ and $\xi, \eta \in S_n$, we have that

\begin{equation}
P(r, \eta) = \sum_{p,q=0}^{\infty} S^{p,q}(r) H^{p,q}(\xi, \eta).
\end{equation}

We need to obtain the expansion of the integral kernel $\Phi_{\alpha}(d(\xi, \eta))$ in terms of these spherical harmonics.

First, fix $\alpha$, with $0 < \alpha < 2n$, and let $\beta = \alpha/4$. If we denote by $g(z)$ the function of one complex variable,

\begin{equation}
g(z) = \frac{1}{|1 - z|^{n/2}} = \frac{1}{|1 - z|^{\beta}}, \quad |z| < 1,
\end{equation}

then we can express the kernel $\Phi_{\eta}(d(\xi, \eta))$ in terms of $g$ as

$$\Phi_{\eta}(d(\xi, \eta)) = g(\langle \xi, \eta \rangle).$$

Now, develop $g(z)$ as a Fourier series in the following way:

**Lemma 1.** For all $n \geq 2$ and $0 < \beta < n/2$, we have the Fourier expansion

$$g(z) = \sum_{p,q=0}^{\infty} g^{p,q} H^{p,q}(z),$$

where $g^{p,q}$ has the expression

\begin{equation}
g^{p,q} = 2^n \frac{\Gamma(n-2\beta)}{\Gamma(\beta)^2} \frac{\Gamma(p+\beta) \Gamma(q+\beta)}{\Gamma(p+n-\beta) \Gamma(q+n-\beta)}.
\end{equation}
In order to prove this result we will need the following

**Lemma 2.** For all $0 \leq \rho < 1$, $\beta > 0$ and all integer $m$, we have that

\[
(2.10) \quad \int_0^{2\pi} \frac{e^{im\theta}}{1 - \rho e^{i\theta}} \, d\theta = 2\pi \rho^{\frac{|m| + \beta}{|m|}} \Gamma(|m| + \beta) \Gamma(\beta) F(\beta, |m| + \beta; |m| + 1; \rho^2).
\]

**Proof of Lemma 2.** Without loss of generality we can assume that $m \geq 0$, since the case $m < 0$ will follow by conjugation. We have that

\[
\int_0^{2\pi} e^{im\theta} (1 - \rho e^{i\theta})^{-3} (1 - \rho e^{-i\theta})^{-3} \, d\theta = \int_0^{2\pi} e^{im\theta} \left( \sum_{k=0}^{\infty} \frac{(\beta)_{k}}{k!} \rho^k e^{ik\theta} \right) \left( \sum_{j=0}^{\infty} \frac{(\beta)_{j}}{j!} \rho^j e^{-ij\theta} \right) \, d\theta
= 2\pi \sum_{k=0}^{\infty} \frac{(\beta)_{k}}{k!} \frac{(-\beta)_{k+m}}{(k+m)!} \rho^{2k+m}
\]

and the lemma follows by substituting the definition of the Pochhammer symbols and using the definition of the hypergeometric function. Q.E.D.

**Proof of Lemma 1.** We will use in this proof the notation $\langle \phi, \psi \rangle$ to denote the usual scalar product in $L^2(\{|z| < 1\})$ with respect to the measure $(1 - |z|^2)^{n-2} \, dx \, dy$. This will not cause confusion since we will not use the inner product in $C^n$ along this proof.

We have that

\[
(2.11) \quad g^{p,q} = \frac{\langle g, H^{p,q} \rangle}{\langle H^{p,q}, H^{p,q} \rangle}.
\]

We recall that [R, p. 15]

\[
(2.12) \quad \int_{S_n} \varphi(\langle \xi, \eta \rangle) \, d\eta = \frac{(n-1) \omega_{2n}}{\pi} \int_0^{2\pi} \int_0^{\pi} \varphi(\rho e^{i\theta}) (1 - \rho^2)^{n-2} \rho \, d\rho \, d\theta,
\]

for all $\varphi \in L^1(\{|z| < 1\})$ with respect to the measure $(1 - |z|^2)^{n-2} \, dx \, dy$. Using (2.5) and (2.12), we deduce that

\[
\frac{D}{\omega_{2n}} = \int_{S_n} |H^{p,q}(\langle \xi, \eta \rangle)|^2 \, d\eta = \frac{(n-1) \omega_{2n}}{\pi} \int_0^{2\pi} \int_0^{\pi} |H^{p,q}(\rho e^{i\theta})|^2 (1 - \rho^2)^{n-2} \rho \, d\rho \, d\theta,
\]

and then

\[
(2.13) \quad \langle H^{p,q}, H^{p,q} \rangle = \frac{\pi D}{(n-1) \omega_{2n}^2}.
\]

On the other hand, since $|p - q| = u - v$,

\[
\langle g, H^{p,q} \rangle = \frac{D}{\omega_{2n} P_{r}^{(n-2, u-v)}(1)} \int_0^{2\pi} \int_0^{\pi} e^{i(\gamma - p)\theta} \rho^{\frac{(n-2, u-v)}{2} - 1} P_{r}^{(n-2, u-v)}(2\rho^2 - 1) \rho^{u-v} (1 - \rho^2)^{n-2} \, d\theta \, d\rho
\]

\[
= \frac{2\pi D}{\omega_{2n} P_{r}^{(n-2, u-v)}(1)} \frac{\Gamma(u - v + \beta)}{(u - v) \Gamma(\beta)} \int_0^{1} F(\beta, u - v + \beta; u - v + 1; \rho^2) P_{r}^{(n-2, u-v)}(2\rho^2 - 1) \rho^{2(u-v)} (1 - \rho^2)^{n-2} \, d\rho
\]

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By making the variable change $t = 2\rho^2 - 1$, we obtain that

$$
\langle g, H^{\rho,\beta} \rangle = \frac{2\pi D}{2^{n-u-v} \omega_{2n}} \frac{\Gamma(u - v + \beta)}{(u - v)! \Gamma(\beta)} \int_{-1}^{1} F(\beta, u - v + \beta; u - v + 1; (1 + t)/2) P_v^{(n-2, u-v)}(t) (1-t)^{n-2}(1+t)^{u-v} \, dt.
$$

If we denote by $(\phi, \psi)$ the scalar product in $L^2[-1, 1]$ with respect to the measure $(1-t)^{n-2}(1+t)^{u-v} \, dt$, then the above formula can be written as

(2.14) $$
\langle g, H^{\rho,\beta} \rangle = \frac{2\pi D}{2^{n-u-v} \omega_{2n}} \frac{\Gamma(u - v + \beta)}{(u - v)! \Gamma(\beta)} (F, P_v^{(n-2, u-v)}) \cdot (F, P_v^{(n-2, u-v)})
$$

where $F$ denotes the hypergeometric function $F(\beta, u - v + \beta; u - v + 1; (1 + t)/2)$.

It is known by [L2, p. 29] that

$$
G(w) = F(a_1, a_2; a_3; (1 + w)t) = \sum_{v=0}^{\infty} C_v P_v^{(a,b)}(w),
$$

where

$$
C_v = \frac{(a_1)_v(a_2)_v(2t)^v}{(a_3)_v(v + \lambda)_v} 3F_2(b + 1 + v, a_1 + v, a_2 + v; \lambda + 1 + 2v, a_3 + v; 2t).
$$

Here $3F_2$ is the following generalized hypergeometric function:

$$
3F_2(a_1, a_2, a_3; \beta_1, \beta_2; t) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k(a_3)_k}{(\beta_1)_k(\beta_2)_k} \frac{t^k}{k!}.
$$

This gives

(2.15) $$
\frac{(G, P_v^{(a,b)})}{(P_v^{(a,b)}, P_v^{(a,b)})} = C_v
$$

for all $t < 1/2$. Then by making $t \to 1/2$, we obtain that (2.15) is also true for $t = 1/2$. Therefore,

$$
(F, P_v^{(n-2, u-v)}) = C_v (P_v^{(n-2, u-v)}, P_v^{(n-2, u-v)})
$$

$$
= \frac{(\beta)_v(u - v + \beta)_v}{(u - v + 1)_v(u + n + 1 - v)_v} 3F_2(u + 1, v + \beta, u + v + n, u + v + n + 1; 1) (P_v^{(n-2, u-v)}, P_v^{(n-2, u-v)})
$$

$$
= \frac{(\beta)_v(u - v + \beta)_v}{(u - v + 1)_v(u + n + 1 - v)_v} F(v + \beta, u + \beta; u + v + n, u + v + n + 1) \frac{2^{u+v+n-1}}{u + v + n - 1} \frac{(u + n - 2)!}{u!}.
$$

where we have used the fact that ([L1, p. 276], [A.S. p. 774])

$$
(P_v^{(a,b)}, P_v^{(a,b)}) = \frac{2^{a+b+1}}{2(a+b+1)!} \frac{\Gamma(v + a + 1) \Gamma(v + b + 1)}{\Gamma(v + a + b + 1)}.
$$

Hence, using Gauss summation formula ([L1, p. 99], [A.S. p. 556]),

$$
F(a, b, c; 1) = \frac{\Gamma(c) \Gamma(c - a - b) \Gamma(c - b)}{\Gamma(c - a) \Gamma(c - b)} \quad \text{if} \quad c - a - b > 0,
$$

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we obtain that
\[
(F, P_v^{(n-2, u-v)}) = 2^{u-v+n-1}(\beta)_v (u - v + \beta)_v \frac{(v + n - 2)! (u - v)! \Gamma(n - 2\beta)}{v! \Gamma(u + n - \beta) \Gamma(v + n - \beta)}.
\]

By substituting this formula (which has sense since \(n - 2\beta > 0\)) in (2.14) we obtain that
\[
\langle g, H^{p,q} \rangle = \frac{(n-2)\pi D \Gamma(n-2\beta)}{\omega_{2n}} \frac{\Gamma(\beta)^2}{\Gamma(n-2\beta)} \frac{\Gamma(u + \beta) \Gamma(v + \beta) \Gamma(u + n - \beta) \Gamma(v + n - \beta)}{\Gamma(p + \beta) \Gamma(q + \beta)}.
\]

(2.16)

where we have used the fact that ([L1, p. 274], [A S, p. 774])
\[
P_v^{(n-2, u-v)}(1) = \frac{(v + n - 2)!}{v! (n - 2)!}.
\]

The lemma follows now by substituting (2.13) and (2.16) in (2.11), and by using that \(\omega_{2n} = 2\pi^n / (n - 1)!\).

Q.E.D.

**Proof of Theorem 2.** We choose an orthonormal basis \(\{ f_{j}^{p,q} \}_{j=1}^{D} \) of \( H^{p,q} \), for each \( p, q \geq 0 \). Let \( \{ \mu_{j}^{p,q} \} \), \( p, q \geq 0 \), \( 1 \leq j \leq D = D(p,q;n) \), be the Fourier coefficients of \( \mu \), i.e.,

\[
\mu \sim \sum_{p,q=0}^{\infty} \sum_{j=1}^{D} \mu_{j}^{p,q} f_{j}^{p,q}.
\]

Recall that \( P_{\mu} \) is defined by
\[
P_{\mu}(r \xi) = \int_{S_{n}} P(r \xi, \eta) d\mu(\eta),
\]

where \( P(r \xi, \eta) \) is the Poisson-Szegö kernel
\[
P(r \xi, \eta) = \frac{1}{\omega_{2n}} \left( \frac{1 - r^2}{1 - r\xi \eta} \right)^{n}, \quad 0 \leq r < 1, \ \xi, \eta \in S_{n}.
\]

Recalling (2.4) and (2.7) we deduce that
\[
P(r \xi, \eta) = \sum_{p,q=0}^{\infty} S^{p,q}(r) H^{p,q}(r, \eta) = \sum_{p,q,j} S^{p,q}(r) f_{j}^{p,q}(r) f_{j}^{p,q}(\eta).
\]

Now, Plancherel’s theorem gives that
\[
P_{\mu}(r \xi) = \sum_{p,q,j} S^{p,q}(r) \mu_{j}^{p,q} f_{j}^{p,q}(\xi).
\]

Using again Plancherel’s theorem we obtain that
\[
\int_{S_{n}} |P_{\mu}(r \xi)|^2 d\xi = \sum_{p,q,j} (S^{p,q}(r))^2 |\mu_{j}^{p,q}|^2,
\]

and so if we denote by \( \Lambda \) the right hand side in (1.4), we have that (recall that \( \beta = \alpha/4 \))
\[
\Lambda = \sum_{p,q,j} |\mu_{j}^{p,q}|^2 \int_{0}^{1} (S^{p,q}(r))^2 r^{2\beta-1} (1 - r^2)^{n-2\beta-1} dr.
\]
and, substituting \( r^2 = t \), we get by using the definition of \( S_{p,q}^\beta(t) \) that

\[
\Lambda = \frac{1}{2} \sum_{p,q,j} \mu_{j}^{p,q} t^{\frac{1}{2}} \int_{0}^{1} \left( \frac{F(p,q; p + q + n; t)}{F(p,q; p + q + n; 1)} \right)^2 t^{p+q+\beta-1}(1-t)^{n-2-\beta-1} dt
\]

(2.17)

\[
\cong \sum_{p,q,j} \frac{\Gamma(p + \beta)}{\Gamma(p + n - \beta)} \sum_{j=1}^{D} \mu_{j}^{p,q} t^{\frac{1}{2}} \mu_{j}^{p,q} \cong g^{p,q} \sum_{j=1}^{D} \mu_{j}^{p,q} t^{\frac{1}{2}} \mu_{j}^{p,q} \cong g^{p,q} \sum_{j=1}^{D} \mu_{j}^{p,q} t^{\frac{1}{2}} \mu_{j}^{p,q}
\]

where we have used Theorem A and Lemma 1.

On the other hand, using again Lemma 1 and (2.4),

\[
\Phi_{\alpha}(d(\xi, \eta)) = g(\langle \xi, \eta \rangle) = \sum_{p,q,j} g^{p,q} H^{p,q}(\langle \xi, \eta \rangle) = \sum_{p,q,j} g^{p,q} f^{p,q}(\xi) f^{p,q}(\eta),
\]

and using Plancherel’s theorem and (2.17), we obtain that

\[
\int_{S_n} \Phi_{\alpha}(d(\xi, \eta)) d\mu(\eta) = \sum_{p,q,j} g^{p,q} \mu_{j}^{p,q} f_{j}^{p,q}(\xi),
\]

\[
J_{\alpha}(\mu) = \sum_{p,q,j} g^{p,q} \mu_{j}^{p,q} \cong \Lambda.
\]

This finishes the proof of Theorem 2. Q.E.D.

3. Proof of Theorem 1.

We need the following lemmas.

**Lemma 3** [FPR]. Let \( \mu \) be a finite positive measure in \( \partial \Delta \), and let \( f \) be an inner function. Then, there exists a unique positive measure \( \nu \) in \( S_n \) such that \( P_{\mu} \circ f = P_{\nu} \) and

\[
\nu(\text{support } \mu) = \nu(S_n).
\]

Moreover, if \( f(0) = 0 \), then

\[
\frac{1}{\omega_{2n}} \nu(S_n) = \frac{1}{2\pi} \mu(\partial \Delta).
\]

A different normalization is useful; choosing \( \nu = (2\pi/\omega_{2n}) \nu \), one obtains

\[
P_{\nu} = \frac{2\pi}{\omega_{2n}} P_{\mu} \circ f \quad \text{and} \quad \nu(S_n) = \mu(\partial \Delta).
\]

The following is well known.
Lemma 4 (Subordination principle). Let $f : B_n \rightarrow \Delta$ be a holomorphic function such that $f(0) = 0$, and let $v : \Delta \rightarrow \mathbb{R}$ be a subharmonic function. Then, for all $0 \leq r < 1$,
\[ \frac{1}{\omega_{2n}} \int_{S_n} v(f(r\xi)) \, d\xi \leq \frac{1}{2\pi} \int_0^{2\pi} v(re^{i\theta}) \, d\theta. \]

Lemma 5. Let $\mu$ be a complex measure on $\partial \Delta$, $f$ an inner function with $f(0) = 0$, and $v$ a complex measure on $S_n$ such that
\[ \mathcal{P}_v = (2\pi/\omega_{2n}) |\mathcal{P}_\mu \circ f|. \]
Then:

i) If $n \geq 1$ and $0 < \alpha < 2$ or $n = 1$ and $\alpha = 0$, there exists a constant $C$ depending at most on $n$ and $\alpha$ such that
\[ J_{2n-2+\alpha}(v) \leq C J_0(\mu). \]

ii) If $\alpha = 0$, $n \geq 2$ and $m = \mu(\partial \Delta)$, there exists a constant $C$ depending at most on $n$ such that
\[ J_{2n-2}(v) \leq C (|m|^2 + J_0(\mu)). \]

Proof. Since $v = |\mathcal{P}_\mu|^2$ is subharmonic (in the euclidean sense), we obtain by subordination (Lemma 4), that
\[(3.1) \quad \int_{S_n} \frac{1}{\omega_{2n}} \int_{S_n} \mathcal{P}_\mu(f(r\xi))^2 \, d\xi \leq \frac{2\pi}{\omega_{2n}} \int_0^{2\pi} |\mathcal{P}_\mu|^2 \, d\theta. \]
Using Theorem 2 twice, the inequality (3.1), and the fact that $n \geq 1$, we have that
\[ J_{2n-2+\alpha}(v) \leq C \int_0^{1} \left\{ \int_{S_n} \mathcal{P}_\mu(f(r\xi))^2 \, d\xi \right\} \frac{dr}{(1 - r^2)^{n/2}} \]
\[ \leq C \int_0^{1} \left\{ \int_0^{2\pi} \mathcal{P}_\mu(re^{i\theta})^2 \, d\theta \right\} \frac{dr}{(1 - r^2)^{n/2}} \]
\[ \leq C \int_0^{1} \left\{ \int_0^{2\pi} \mathcal{P}_\mu(re^{i\theta})^2 \, d\theta \right\} \frac{dr}{(1 - r^2)^{n/2}} \]
\[ \leq C J_0(\mu). \]
This finishes the proof of part i) in the case $n \geq 1$, $0 < \alpha < 2$. The other case follows from [FPR, Lemma 5], since $J_0(v) = I_0(v)/2$.

In order to prove ii), using that $m = \mu(\partial \Delta) = \nu(S_n)$, we obtain that
\[ \int_{S_n} \mathcal{P}_\nu(r\xi) - \frac{m^2}{\omega_{2n}} \, d\xi + \frac{|m|^2}{\omega_{2n}} = \int_{S_n} \mathcal{P}_\nu(r\xi)^2 \, d\xi. \]
Integrating this equality and using Theorem 2, we have that
\[ J_{2n-2}(v) \leq C \int_0^{1} \int_{S_n} \mathcal{P}_\nu(r\xi) - \frac{m^2}{\omega_{2n}} \, d\xi \, r^{n-2} \, dr + \frac{|m|^2}{(n-1)\omega_{2n}} \]
\[ \leq C \int_0^{1} \int_{S_n} \mathcal{P}_\mu(re^{i\theta}) - \frac{m^2}{2\pi} \, d\theta \, r \, dr + \frac{|m|^2}{(n-1)\omega_{2n}} \]
\[ \leq C (|m|^2 + J_0(\mu)) \leq C (|m|^2 + J_0(\mu)). \]
where we have used subordination (Lemma 4) with $v = |\mathcal{P}_\mu - m/(2\pi)|^2$ and Theorem B. Q.E.D.

Finally we can finish the proof of Theorem 1. We may assume that $E$ is closed. In order to prove (1.1), let us denote by $\mu_e$ the $\alpha$-equilibrium probability distribution of $E$ and let $\nu$ be the probability measure in $S_n$ such that $\mathcal{P}_\nu = (2\pi/\omega_{2n}) \mathcal{P}_{\mu_e} \circ f$. By Lemma 5,

\[(3.2) \quad J_{2n-2+\alpha}(\nu) \leq C J_{\alpha}(\mu_e) = C (\text{cap}_\alpha(E))^{-1}.
\]

But, from Lemma 3, $\nu(f^{-1}(E)) = 1$, and so

\[J_{2n-2+\alpha}(\nu) = \int_{f^{-1}(E) \times f^{-1}(E)} \Phi_{2n-2+\alpha}(d(\xi, \eta)) \, d\nu(\xi) \, d\nu(\eta).
\]

Now, let $\{K_j\}$ be an increasing sequence of compact subsets of $f^{-1}(E)$, such that $\nu(K_j) \nearrow 1$. Then, for each $j$,

\[J_{2n-2+\alpha}(\nu) = \int_{f^{-1}(E) \times f^{-1}(E)} \Phi_{2n-2+\alpha}(d(\xi, \eta)) \, d\nu(\xi) \, d\nu(\eta)
\]

\[\geq \nu(K_j)^2 \int_{K_j \times K_j} \Phi_{2n-2+\alpha}(d(\xi, \eta)) \, d\nu(\xi) \, d\nu(\eta)
\]

\[\geq \nu(K_j)^2 \left( \text{cap}_{2n-2+\alpha}(K_j) \right)^{-1}
\]

and consequently, if we let $j \to \infty$, we obtain that

\[(3.3) \quad J_{2n-2+\alpha}(\nu) \geq \left( \text{cap}_{2n-2+\alpha}(f^{-1}(E)) \right)^{-1}.
\]

Therefore, in the case $0 < \alpha < 2$, $n \geq 1$, (1.1) follows now from (3.2) and (3.3). The case $\alpha = 0$, $n = 1$, follows from [FPR, Theorem 1].

In order to prove (1.2) we proceed as follows. Let $\mu_e$ be the equilibrium distribution of $E$ for the logarithmic capacity and let $\nu$ be the measure supported on $S_n$ such that $\mathcal{P}_\nu = (2\pi/\omega_{2n}) \mathcal{P}_{\mu_e} \circ f$. Using Lemma 5,

\[J_{2n-2}(\nu) \leq C (1 + J_0(\mu_e)) = C \left( 1 + \log \frac{1}{\text{cap}_0(E)} \right).
\]

Now, to finish the proof, one needs only to follow the same lines that in part i).

References.


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