LOCATION OF GEODESICS AND ISOPERIMETRIC INEQUALITIES IN DENJOY DOMAINS

JOSÉ M. RODRÍGUEZ† AND JOSÉ M. SIGARRETA∗

ABSTRACT. We find approximate solutions (chord-arc curves) for the system of equations of geodesics in \( \Omega \cap \mathbb{H} \) for every Denjoy domain \( \Omega \), with respect to both the Poincaré and the quasihyperbolic metrics. We also prove that these chord-arc curves are uniformly close to the geodesics. As an application of these results, we improve the characterization in [2] of the Denjoy domains satisfying the linear isoperimetric inequality.

1. Introduction

In the current paper our main aim is to study the geodesics of Denjoy domains, that is to say, plane domains \( \Omega \) with \( \partial \Omega \subset \mathbb{R} \). This kind of surfaces are becoming more and more important in Geometric Theory of Functions, since, on the one hand, they are a very general type of Riemann surfaces, and, on the other hand, they are more manageable due to its symmetry. For instance, Garnett and Jones have proved in [15] the Corona Theorem for Denjoy domains, and in [2] the authors have got the characterization of Denjoy domains which satisfy a linear isoperimetric inequality.

To know explicitly the location of the geodesics in a Riemannian surface is not possible except for a few examples, since in order to do it we must solve a second order system of two non-linear differential equations. In the case of a domain with the Poincaré or the quasihyperbolic metric, the situation is even worse: on the one hand, usually we do not have an explicit expression for the density of the Poincaré metric, and hence for the equations; on the other hand, for the quasihyperbolic metric the coefficients in the differential equations are the derivatives of a non-differentiable function. However, the geodesics are a main object in Riemannian geometry.

In this paper we find approximate solutions (chord-arc curves, a very regular kind of quasigeodesics, see Definition 2.3) for the system of equations of geodesics in \( \Omega \cap \mathbb{H} \) for every Denjoy domain \( \Omega \) (see Theorems 4.2 and 4.4). Furthermore, using results on Gromov hyperbolicity (although in general \( \Omega \) is not Gromov hyperbolic!), we also prove that these chord-arc curves are uniformly close to the geodesics (see Theorems 4.5 and 4.7). There are several papers studying Gromov hyperbolicity of Euclidean domains and Riemann surfaces in general, and Denjoy domains in particular (see [3], [5], [8], [19]–[25] and [30]–[37]; see also [9], [39] and [40]).
Using these results on chord-arc curves we obtain good estimates for the Poincaré distance of:

(i) any couple of points \( z, w \in \Omega \cap \mathbb{R} \) (see Theorem 5.3),

(ii) any couple of connected components of \( \Omega \cap \mathbb{R} \) (see Theorem 5.4),

(iii) any point \( z \in \Omega \cap \mathbb{R} \) and any connected component of \( \Omega \cap \mathbb{R} \) (see Theorem 5.6).

In particular, (ii) is equivalent to estimate the length of simple closed geodesics, a very interesting and difficult problem for the Poincaré metric.

We obtain these estimates up to multiplicative constants, which are the best possible results for the Poincaré metric, since the sharpest known estimate for the density of the Poincaré metric (see Theorem 2.8) also has this property.

In [21] there is a weaker version of Theorems 4.2 and 4.4, but in [21] the authors proved that the curves are \((a, b)\)-quasigeodesics with \( b > 0 \); this additive constant \( b \) is a “large error” in order to deal with some applications, see Remark 6.8 (although it is good enough for the purposes in [21]).

As an application of these results, we improve the characterization in [2] of the Denjoy domains satisfying the linear isoperimetric inequality (see Theorem 6.9).

**Notations.** If we do not specify the metric, we always assume that in any Denjoy domain \( \Omega \) we consider the Poincaré metric. By \( d_\Omega, L_\Omega \) and \( A_\Omega \) we shall denote, respectively, the distance, the length and the area with respect to the Poincaré metric of \( \Omega \).

**Acknowledgements.** We would like to thank Venancio Alvarez for several useful comments.

### 2. Previous definitions and results

We denote by \( \mathbb{H} \) the upper half plane, \( \{ z \in \mathbb{C} : \, \text{Im} \, z > 0 \} \) and by \( \mathbb{D} \) the unit disk \( \{ z \in \mathbb{C} : \, |z| < 1 \} \). For \( D \subset \mathbb{C} \) we denote by \( \partial D \) and \( \overline{D} \) its boundary and closure, respectively. For \( z \in D \subset \mathbb{C} \) we denote by \( \delta_D(z) \) the distance to the boundary, \( \min_{a \in \partial D} |z - a| \).

The **quasihyperbolic metric** is the distance induced by the density \( 1/\delta_\Omega(z) \).

Recall that a domain \( \Omega \subset \mathbb{C} \) is said to be of **non-exceptional** if it has at least two finite boundary points. The universal cover of such domain is the unit disk \( \mathbb{D} \). In \( \Omega \) we can define the Poincaré metric, i.e. the metric obtained by projecting the metric \( ds = 2|dz|/|1 - |z|^2| \) of the unit disk by any universal covering map \( \pi : \mathbb{D} \longrightarrow \Omega \). Equivalently, we can project the metric \( ds = |dz|/\text{Im} \, z \) of the upper half plane \( \mathbb{H} \). Therefore, any simply connected subset of \( \Omega \) is isometric to a subset of \( \mathbb{D} \). With this metric, \( \Omega \) is a geodesically complete Riemannian manifold with constant curvature \(-1\); in particular, \( \Omega \) is a geodesic metric space. The **Poincaré metric** is natural and useful in complex analysis; for instance, any holomorphic function between two domains is Lipschitz with constant \( 1 \), when we consider the respective Poincaré metrics.

We denote by \( \lambda_\Omega \) the density of the hyperbolic metric in \( \Omega \). It is well known that for all domains \( \Omega_1 \subseteq \Omega_2 \) we have \( \lambda_{\Omega_1}(z) \geq \lambda_{\Omega_2}(z) \) for every \( z \in \Omega_1 \).

A **Denjoy domain** \( \Omega \subset \mathbb{C} \) is a domain whose boundary is contained in the real axis. As we mentioned in the introduction of this paper, Denjoy domains are becoming more and more interesting in Geometric Function Theory (see e.g. [1], [2], [15], [17]).
Definition 2.1. If $\gamma : [a, b] \rightarrow X$ is a continuous curve in a metric space $(X, d)$, the length of $\gamma$ is

$$L(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \cdots < t_n = b \right\}.$$ 

We say that $\gamma$ is a geodesic if it is an isometry, i.e., $L(\gamma|_{[t, s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$. We say that $X$ is a geodesic metric space if for every $x, y \in X$ there exists a geodesic joining $x$ and $y$; we denote by $xy$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but convenient as well).

Definition 2.2. Consider a geodesic metric space $X$. If $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of three geodesics $x_1x_2, x_2x_3$ and $x_3x_1$. We say that $T$ is $\delta$-thin if for every $x \in x_1x_j$ we have that $d(x, x_jx_k \cup x_kx_i) \leq \delta$. The space $X$ is $\delta$-hyperbolic (or satisfies the Rips condition with constant $\delta$) if every geodesic triangle in $X$ is $\delta$-thin.

Examples:

1. Every bounded metric space $X$ is $(\text{diam}X)$-hyperbolic (see e.g. [16, p. 29]).
2. Every complete simply connected Riemannian manifold with sectional curvature $\leq -k$, with $k > 0$, is hyperbolic (see e.g. [16, p. 52]).
3. Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [16, p. 29]).

Definition 2.3. A function between two metric spaces $f : X \rightarrow Y$ is an $(a, b)$-quasi-isometry, $a \geq 1$, $b \geq 0$, if

$$\frac{1}{a} d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a d_X(x_1, x_2) + b,$$

for every $x_1, x_2 \in X$.

An $(a, b)$-quasigeodesic in $X$ is an $(a, b)$-quasi-isometry between an interval of $\mathbb{R}$ and $X$.

A map $f$ between an interval $I$ of $\mathbb{R}$ and $X$ is $a$-chord-arc if

$$L_X(f|_{[x_1, x_2]}) \leq a d_X(x_1, x_2), \quad \text{for every } [x_1, x_2] \subseteq I.$$

Chord-arc curves play a main role in harmonic analysis and in geometry. It is clear that the $a$-chord-arc curves with their arc-length parametrization are $(a, 0)$-quasigeodesics; they are a very special type of “very regular” quasigeodesics (note that a quasigeodesic can be discontinuous).

Definition 2.4. Let us consider $\varepsilon > 0$, a metric space $X$, and subsets $Y, Z \subseteq X$. The set $N_\varepsilon(Y) := \{x \in X : d(x, Y) \leq \varepsilon\}$ is called the $\varepsilon$-neighborhood of $Y$ in $X$. The Hausdorff distance of $Y$ to $Z$ is defined by $H(Y, Z) := \inf\{\varepsilon > 0 : Y \subseteq N_\varepsilon(Z), Z \subseteq N_\varepsilon(Y)\}$.

The following is a beautiful and useful result:

Theorem 2.5. ([16, p.87]) For each $\delta \geq 0$, $a \geq 1$ and $b \geq 0$, there exists a constant $H_0$, which just depends on $\delta$, $a$, and $b$, with the following property:

Let us consider a $\delta$-hyperbolic geodesic metric space $X$ and an $(a, b)$-quasigeodesic $g$ starting in $x$ and finishing in $y$. If $\gamma$ is a geodesic joining $x$ and $y$, then $H(g, \gamma) \leq H_0$.

This property is known as geodesic stability. Mario Bonk has proved that, in fact, geodesic stability is equivalent to hyperbolicity (see [7]). There is an explicit expression for $H_0$, but it is very complicated; however, we have the following particular result which gives a simple bound.
**Theorem 2.6.** ([7, Proposition 3.1]) Let $X$ be a $\delta$-hyperbolic geodesic metric space and $g$ an $a$-chord-arc curve joining $x$ and $y$. Then there exists a constant $M$, which just depends on $\delta$ and $a$, such that $g \subset N_M(\gamma)$ for every geodesic $\gamma$ joining $x$ and $y$.

Furthermore, we can take

$$M = M(\delta, a) := (1 + 8\delta a)(8\delta a^2 + 12\delta a + 2a) + 4\delta a + 2\delta + 2.$$ 

**Definition 2.7.** For every non-exceptional domain $\Omega \subset \mathbb{C}$ and for every $z \in \Omega$, define $\delta_{\Omega}(z) := \inf\{|z - a| : a \in \partial \Omega\}$ and $\beta_{\Omega}(z)$ as the function

$$\beta_{\Omega}(z) := \inf \left\{ \left| \log \frac{|z - a|}{|b - a|} \right| : a, b \in \partial \Omega, |z - a| = \delta_{\Omega}(z) \right\}.$$ 

It is clear that the infimum in $\delta_{\Omega}(z)$ and in $\beta_{\Omega}(z)$ is attained.

The function $\beta_{\Omega}$ was introduced by Beardon and Pommerenke [6] who showed that it provides the connection between the densities of the hyperbolic and the quasihyperbolic metrics.

**Theorem 2.8.** ([6, Theorem 1]) For every non-exceptional domain $\Omega \subset \mathbb{C}$ and for every $z \in \Omega$, we have that

$$2^{-3/2} \leq \lambda_{\Omega}(z) \delta_{\Omega}(z) (k_0 + \beta_{\Omega}(z)) \leq \pi/4,$$

where $k_0 = 4 + \log(3 + 2\sqrt{2})$.

It follows from this theorem, that $\lambda_{\Omega}$ and $1/\delta_{\Omega}$ are comparable if and only if $\mathbb{C} \setminus \Omega$ is uniformly perfect (see [6], [28], [29]).

3. Technical lemmas

In this section some technical lemmas are collected. All of them have been used in the next section in order to simplify the proof of Theorem 4.2.

**Definition 3.1.** For $c \geq 1$, let us define the function

$$k(c) := \frac{c \pi}{\sqrt{2}} \left( 1 + \frac{1}{k_0} \log c \right).$$

**Lemma 3.2.** Let us consider any non-exceptional domain $\Omega$, $z \in \Omega$, $a \in \partial \Omega$ with $|z - a| = \delta_{\Omega}(z)$, and $c \geq 1$. For every $w \in \Omega$ with $|w - a| \leq c|z - a|$, we have $\lambda_{\Omega}(z) \leq k(c) \lambda_{\Omega}(w)$.

**Remark 3.3.** This result trivially holds for the quasihyperbolic metric, replacing $k(c)$ by $c$, since

$$\frac{1}{\delta_{\Omega}(z)} = \frac{1}{|z - a|} \leq \frac{c}{|w - a|} \leq \frac{c}{\delta_{\Omega}(w)}.$$ 

This remark is applicable to every Lemma in this section.

**Proof.** Let us assume first that there exists $b \in \partial \Omega$ with

$$\beta_{\Omega}(z) = \left| \log \frac{|z - a|}{|b - a|} \right|.$$ 

(Although the infimum in $\beta_{\Omega}(z)$ is attained, perhaps $\beta_{\Omega}(z) = \left| \log |z - a'|/|b' - a'| \right|$ with $a' \neq a$ and $|z - a'| = |z - a| = \delta_{\Omega}(z)$.)
Assume now that $|w - a| \leq |z - a|$. Since the function $f(x) := x (k_0 + |\log(x/s)|)$ is increasing in $x \in (0, \infty)$ for any fixed positive constant $s$, then

$$
\lambda_\Omega(z) \leq \frac{\pi/4}{|z - a| \left( k_0 + \left| \log \left| \frac{z - a}{b - a} \right| \right) } \leq \frac{\pi/4}{w - a| \left( k_0 + \left| \log \left| \frac{w - a}{b - a} \right| \right) }.
$$

Note that if $\Omega_0 := \mathbb{C \setminus \{a, b\}}$ and $|w - b| < |w - a|$, then

$$
\frac{|w - a| \left( k_0 + \left| \log \left| \frac{w - a}{b - a} \right| \right) }{2^{-3/2}} \leq \frac{|w - b| \left( k_0 + \left| \log \left| \frac{w - b}{b - a} \right| \right) }{2^{-3/2}} = \frac{\delta_{\Omega_0}(w) \left( k_0 + \left| \log \left| \frac{\delta_{\Omega_0}(w)}{b - a} \right| \right) }{\delta_{\Omega_0}(w) \left( k_0 + \left| \log \left| \frac{\delta_{\Omega_0}(w)}{b - a} \right| \right) } \leq \lambda_{\Omega_0}(w).
$$

If $|w - b| \geq |w - a|$, we also have

$$
\frac{|w - a| \left( k_0 + \left| \log \left| \frac{w - a}{b - a} \right| \right) }{2^{-3/2}} = \frac{\delta_{\Omega_0}(w) \left( k_0 + \left| \log \left| \frac{\delta_{\Omega_0}(w)}{b - a} \right| \right) }{\delta_{\Omega_0}(w) \left( k_0 + \left| \log \left| \frac{\delta_{\Omega_0}(w)}{b - a} \right| \right) } \leq \lambda_{\Omega_0}(w).
$$

Consequently, in both cases,

$$
\lambda_\Omega(z) \leq \frac{\pi/4}{|w - a| \left( k_0 + \left| \log \left| \frac{w - a}{b - a} \right| \right) } \leq 2^{3/2} \frac{\pi/4}{\lambda_{\Omega_0}(w)} \leq 2^{-1/2}\pi \lambda_\Omega(w).
$$

Assume now that $|z - a| < |w - a| \leq c|z - a|$. Note that for any $u, v \in \mathbb{R}$ we have

$$
\frac{k_0 + |u|}{k_0 + |u - v|} \leq 1 + \frac{|v|}{k_0}.
$$

Hence

$$
\lambda_\Omega(z) \leq \frac{\pi/4}{|z - a| \left( k_0 + \left| \log \left| \frac{z - a}{b - a} \right| \right) } \leq \frac{\pi/4}{\frac{1}{c} |w - a| \left( k_0 + \left| \log \frac{1}{c} \left| \frac{w - a}{b - a} \right| \right) } \leq \frac{\frac{\pi/4}{k_0 + \left| \log \left| \frac{w - a}{b - a} \right| \right| - \log c}{k_0 + \left| \log \left| \frac{w - a}{b - a} \right| \right|} \cdot \frac{c \pi/4}{|w - a| \left( k_0 + \left| \log \left| \frac{w - a}{b - a} \right| \right) } \leq \left( 1 + \frac{1}{k_0} \log c \right) \frac{2^{-3/2} c \pi/\sqrt{2}}{|w - a| \left( k_0 + \left| \log \left| \frac{w - a}{b - a} \right| \right) } \leq \frac{2^{-3/2} k(c)}{|w - a| \left( k_0 + \left| \log \left| \frac{w - a}{b - a} \right| \right) }.
$$

As in the previous case,

$$
\frac{2^{-3/2}}{|w - a| \left( k_0 + \left| \log \left| \frac{w - a}{b - a} \right| \right) } \leq \lambda_{\Omega_0}(w),
$$
and then
\[ \lambda_\Omega(z) \leq \frac{2^{-3/2}k(c)}{|w-a| \left( k_0 + \log \left( \frac{|w-a|}{b-a} \right) \right)} \leq k(c) \lambda_{\Omega_0}(w) \leq k(c) \lambda_\Omega(w). \]

Let us assume now that there is no \( b \in \partial \Omega \) with
\[ \beta_\Omega(z) = \left| \log \frac{z-a}{b-a} \right|. \]
Without loss of generality we can assume that \( a = 0 \) and \( z > 0 \). For \( 0 < \varepsilon < z \), we have \( \delta_\Omega(z-\varepsilon) = z-\varepsilon \) and \( |z-\varepsilon-\zeta| > \delta_\Omega(z-\varepsilon) \) for every \( \zeta \in \partial \Omega \setminus \{0\} \). Hence, there exists \( b \in \partial \Omega \) with
\[ \beta_\Omega(z-\varepsilon) = \left| \log \frac{z-\varepsilon}{b} \right|. \]
Therefore \( |w| \leq c \varepsilon |z-\varepsilon| \), for some constant \( c \) with \( c \to c \) as \( \varepsilon \to 0 \). Then the Theorem follows by the previous case, since \( \lambda_\Omega \) and \( k(c) \) are continuous functions.

**Lemma 3.4.** Let us consider any non-exceptional domain \( \Omega \) and two curves \( \sigma, \eta \) in \( \Omega \) with the same Euclidean length and parametrized with Euclidean arc-length. Assume that there exists a constant \( c \geq 1 \) with the following property: for each fixed \( t \), there exists \( a_t \in \partial \Omega \) with \( |\sigma(t) - a_t| = \delta_\Omega(\sigma(t)) \) and \( |\eta(t) - a_t| \leq c |\sigma(t) - a_t| \). Then \( L_\sigma(\sigma) \leq k(c) L_\Omega(\eta) \).

**Proof.** Lemma 3.2 gives that \( \lambda_\Omega(\sigma(t)) \leq k(c) \lambda_\Omega(\eta(t)) \) for every \( t \). Since \( \eta(t) \) and \( \sigma(t) \) are parametrized with Euclidean arc-length, this inequality gives \( L_\sigma(\sigma) \leq k(c) L_\Omega(\eta) \).

Using Lemma 3.4 (with \( \sigma(t) = z_0 + it, t \in [0, r] \) and \( c = \sqrt{2} \)), we obtain the following result.

**Lemma 3.5.** Let us consider a Denjoy domain \( \Omega, z_0 \in \Omega \cap \mathbb{H} \), a curve \( \sigma \) with Euclidean length \( r \) starting at \( z_0 \), and \( \sigma := [z_0, z_0 + ir] \). Then \( L_\Omega(\sigma) \leq k(\sqrt{2}) L_\Omega(\eta) \).

**Lemma 3.6.** Let us consider a Denjoy domain \( \Omega, z_0 \in \Omega \) with \( \text{Im} z_0 \geq T > 0 \), a curve \( \eta \) with Euclidean length \( T \) starting at \( z_0 \), and \( \sigma := [z_0, z_0 + T] \). Then \( L_\Omega(\sigma) \leq k(3) L_\Omega(\eta) \).

**Proof.** Consider the curve \( \eta \) parametrized with Euclidean arc-length starting at \( z_0 \). For each fixed \( t \in [0, T] \), let us define \( \sigma(t) := z_0 + t \), and consider \( a_t \in \partial \Omega \) with \( |\sigma(t) - a_t| = \delta_\Omega(\sigma(t)) \). We have
\[ |\eta(t) - a_t| \leq |\eta(t) - z_0| + |z_0 - \sigma(t)| + |\sigma(t) - a_t| \leq 2t + |\sigma(t) - a_t|. \]
Since \( t \leq T \leq \text{Im} z_0 \leq |\sigma(t) - a_t| \), we deduce that
\[ |\eta(t) - a_t| \leq 3 |\sigma(t) - a_t|. \]
Lemma 3.4 with \( c = 3 \) gives the result.

**Definition 3.7.** Let us define the function \( F : \mathbb{C} \to \mathbb{C} \) as
\[ F(re^{it}) := \begin{cases} r + ir \tan t, & \text{if } r \geq 0, \ 0 \leq t \leq \pi/4, \\ r \cotan t + ir, & \text{if } r \geq 0, \ \pi/4 \leq t \leq \pi/2, \end{cases} \]
\[ F(-z) = -F(z) \text{ and } F(\overline{z}) = \overline{F(z)} \text{ for every } z \in \mathbb{C}. \]
Note that the transformation $F$ has a simple geometric meaning: the image by $F$ of the circle $\{ |z| = r \}$ is the boundary of the square $[-r,r] \times [-r,r]$ (i.e., $F$ applies $C$-lines on $B$-lines, see Definition 4.1). This function will allow to obtain information about $C$-lines from results about $B$-lines (see the proof of Theorem 4.2).

**Lemma 3.8.** This function $F$ satisfies
\[
\frac{1}{\sqrt{2}} |z - x| \leq |F(z) - x| \leq \sqrt{3} |z - x|,
\]
for every $z \in \mathbb{C}$ and every $x \in \mathbb{R}$.

**Proof.** By symmetry, it suffices to prove the inequalities for $z = re^{it}$ with $t \in [0,\pi/2]$. Since $F$ is an homogeneous function, it is sufficient to prove the result for $z = e^{it}$ with $t \in [0,\pi/2]$.

We prove just the first inequality, since the proof of the second one uses similar arguments. In order to prove it for $z = e^{it}$ with $t \in [0,\pi/4]$, the following inequalities are equivalent:
\[
|z - x|^2 \leq 2 |F(z) - x|^2,
\]
\[
2x^2 - 4x + 2\sec^2 t \geq x^2 - 2x\cos t + 1,
\]
\[
x^2 - 2(2 - \cos t)x + 2\sec^2 t - 1 \geq 0.
\]
This last inequality is equivalent to $-\cos^4 t + 4\cos^3 t - 5\cos^2 t + 2 \geq 0$ for every $t \in [0,\pi/4]$, and this is equivalent to $f(u) := -u^4 + 4u^3 - 5u^2 + 2 \geq 0$ for every $u \in [1/\sqrt{2},1]$.

Since $f'(u) = -2u(2u^2 - 6u + 5) < 0$ for every $u \in [1/\sqrt{2},1]$, $f$ is decreasing in $[1/\sqrt{2},1]$; hence $f(u) \geq f(1) = 0$ for every $u \in [1/\sqrt{2},1]$.

In order to prove the first inequality of the Lemma for $z = e^{it}$ with $t \in [\pi/4,\pi/2]$, the following inequalities are equivalent:
\[
|z - x|^2 \leq 2 |F(z) - x|^2,
\]
\[
2x^2 - 4x\cot t + 2 + 2\cot^2 t \geq x^2 - 2x\cos t + 1,
\]
\[
x^2 - 2(2\cot t - \cos t)x + 2\cot^2 t + 1 \geq 0.
\]
This last inequality is equivalent to $\sin^4 t - 4\sin^3 t + 2\sin^2 t + 4\sin t - 2 \geq 0$ for every $t \in [\pi/4,\pi/2]$, and this is equivalent to $g(u) := u^4 - 4u^3 + 2u^2 + 4u - 2 \geq 0$ for every $u \in [1/\sqrt{2},1]$.

Since $g''(u) = 24u - 24 \leq 0$ for every $u \in [1/\sqrt{2},1]$, we deduce that $g''(u) = 12u^2 - 24u + 4 \leq g''(1/\sqrt{2}) = 10 - 12\sqrt{2} < 0$ for every $u \in [1/\sqrt{2},1]$. Consequently, $g'(1) = 4u^3 - 12u^2 + 4u + 4 \geq g'(1) = 0$ for every $u \in [1/\sqrt{2},1]$, and $g$ is increasing in $[1/\sqrt{2},1]$; hence $g(u) \geq g(1/\sqrt{2}) = \sqrt{2} - 3/4 > 0$ for every $u \in [1/\sqrt{2},1]$. \hfill \Box

**Lemma 3.9.** The following inequalities hold for the function $F$ and every Denjoy domain $\Omega$:

1. For every $z \in \Omega$,
\[
\frac{1}{k(\sqrt{2})} \lambda_{\Omega}(F(z)) \leq \lambda_{\Omega}(z) \leq k(\sqrt{3}) \lambda_{\Omega}(F(z)).
\]

2. For every curve $\gamma$ contained in any circle $\{|z| = r\} \cap \Omega$,
\[
L_{\Omega}(\gamma) \leq k(\sqrt{3}) L_{\Omega}(F(\gamma)).
\]
(3) For every curve \( g \) contained in \( \Omega \),
\[
L_\Omega(F(g)) \leq 2\sqrt{2}k(\sqrt{2})L_\Omega(g).
\]

(4) For every \( z_1, z_2 \in \Omega \),
\[
d_\Omega(F(z_1), F(z_2)) \leq 2\sqrt{2}k(\sqrt{2})d_\Omega(z_1, z_2).
\]

**Proof.** Note that \( F(\Omega) = \Omega \) for every Denjoy domain \( \Omega \), since \( F(r) = r \) for every \( r \in \mathbb{R} \).

We will prove (2) just for any curve \( \gamma \) contained in \( \{re^{it} \in \mathbb{C} : t \in [0, \pi/4]\} \cap \Omega \), since the other cases are similar. Then \( \gamma(t) = re^{it} \) with \( t \in [t_1, t_2] \subseteq [0, \pi/4] \), and
\[
L_\Omega(\gamma) = \int_\gamma \lambda_\Omega(z) |dz| = \int_{t_1}^{t_2} \lambda_\Omega(re^{it}) r \, dt.
\]

Since \( F(\gamma(t)) = F(re^{it}) = r + ir \tan t \) with \( t \in [t_1, t_2] \), then \( (F(\gamma(t)))' = ir \sec^2 t \). Using part (1) above we obtain
\[
L_\Omega(F(\gamma)) = \int_{F(\gamma)} \lambda_\Omega(z) |dz| = \int_{t_1}^{t_2} \lambda_\Omega(F(re^{it})) r \sec^2 t \, dt
\geq \int_{t_1}^{t_2} \lambda_\Omega(F(re^{it})) r \, dt \geq \frac{1}{k(\sqrt{3})} \int_{t_1}^{t_2} \lambda_\Omega(re^{it}) r \, dt = \frac{1}{k(\sqrt{3})} L_\Omega(\gamma).
\]

We will prove (3) just for any curve \( g \) contained in \( \{re^{it} \in \mathbb{C} : t \in [0, \pi/4]\} \cap \Omega \), since the other cases are similar. Let us consider a parametrization \( g(s) := r(s)e^{i\theta(s)} \) with \( s \in [a, b] \). Then
\[
L_\Omega(g) = \int_a^b \sqrt{r'(s)^2 + r(s)^2\theta'(s)^2} \lambda_\Omega(r(s)e^{i\theta(s)}) \, ds.
\]

Since \( F(g(s)) = r(s) + ir(s) \tan \theta(s) = x(s) + iy(s) \) with \( s \in [a, b] \),
\[
x'(s) = r'(s) + r(s)^2\tan \theta(s) \theta'(s) \sec^2 \theta(s)
\]
\[
y'(s) = r'(s) + r(s)^2\tan \theta(s) \sec^2 \theta(s)
\]
\[
x'(s)^2 + y'(s)^2 \leq r'(s)^2 + 2(r'(s)^2\tan^2 \theta(s) + r(s)^2\theta'(s)^2\sec^4 \theta(s))
\]
\[
\leq r'(s)^2 + 2r'(s)^2 + 8r(s)^2\theta'(s)^2
\]
\[
\leq 8r'(s)^2 + r(s)^2\theta'(s)^2,
\]
and using part (1) above we obtain
\[
L_\Omega(F(g)) = \int_a^b \sqrt{x'(s)^2 + y'(s)^2} \lambda_\Omega(F(r(s)e^{i\theta(s)})) \, ds
\leq 2\sqrt{2} \int_a^b \sqrt{r'(s)^2 + r(s)^2\theta'(s)^2} \lambda_\Omega(F(r(s)e^{i\theta(s)})) \, ds
\leq 2\sqrt{2}k(\sqrt{2}) \int_a^b \sqrt{r'(s)^2 + r(s)^2\theta'(s)^2} \lambda_\Omega(r(s)e^{i\theta(s)}) \, ds = 2\sqrt{2}k(\sqrt{2})L_\Omega(g).
\]

Part (4) follows directly from part (3), by taking \( g \) as a geodesic joining \( z_1 \) and \( z_2 \). \( \Box \)
4. CHORD-ARC CURVES IN EVERY DENJOY DOMAIN

The following curves will play a main role in our results.

**Definition 4.1.** We denote by **A-lines** the set of curves which can be written as \( \{ z \in \mathbb{H} \cap \Omega : \text{Im}\ z = a \} \) for some constant \( a \in \mathbb{R} \).

We denote by **B-lines** the set of curves which can be written as \( ([a, a + ir] \cup [a + ir, a + 2r + ir] \cup [a + 2r + ir, a + 2r]) \cap \Omega \) for some constants \( a \in \mathbb{R}, r > 0 \).

Halfcircles of the type \( \{ z \in \mathbb{H} \cap \Omega : |z - x_0| = r \}, x_0 \in \mathbb{R}, r > 0 \), are called **C-lines**.

Note that **A-lines** and **C-lines** are the geodesics for the Poincaré metric in \( \mathbb{H} \) (and also for the quasihyperbolic metric, since both metrics are the same in \( \mathbb{H} \)). It is useful to consider **B-lines**, since in practical cases the computations with **B-lines** are easier than with the **C-lines**.

The following surprising result says that the geodesics for \( \mathbb{H} \) are chord-arc curves in every Denjoy domain (with universal constants), no matter wether or not some of the endpoints of the curves belongs to \( \partial \Omega \).

**Theorem 4.2.** Let \( \Omega \) be any Denjoy domain. Then the following result holds for the Poincaré metric:

1. Every **A-line** is \( k(\sqrt{2}) \)-chord-arc.
2. Every **B-line** is \( k_1  \)-chord-arc, with \( k_1 := k(\sqrt{2}) + k(3) \).
3. Every **C-line** is \( k_2 \)-chord-arc, with \( k_2 := 2\sqrt{2}k(\sqrt{2})k(\sqrt{3})k_1 \).

**Remark 4.3.** By symmetry, a similar result holds for \( \{ z \in \Omega : \text{Im}\ z \leq 0 \} \).

**Proof.** Consider \( \sigma \), which is either an **A-line**, a **B-line** or a **C-line** parametrized with Poincaré arc-length, and \( s < t \) in the domain of \( \sigma \).

Assume first that \( \sigma \) is an **A-line** \( \sigma = \{ z \in \mathbb{H} \cap \Omega : \text{Im}\ z = a \} \). Let us consider a hyperbolic geodesic \( \eta \) joining \( \sigma(s) \) and \( \sigma(t) \). Without loss of generality we can assume that \( \text{Im}\ \sigma(s) < \text{Im}\ \sigma(t) \). Since the graph of \( \sigma \) is a straight line, we obtain \( L_{\text{Eucl}}(\sigma_{[s,t]}) \leq L_{\text{Eucl}}(\eta) \), and we can denote by \( \eta_0 \) the subcurve of \( \eta \) starting at \( \sigma(s) \) with \( L_{\text{Eucl}}(\eta_0) = L_{\text{Eucl}}(\sigma_{[s,t]}) \). Applying Lemma 3.5 we deduce

\[
 t - s = L_{\Omega}(\sigma_{[s,t]}) \leq k(\sqrt{2})L_{\Omega}(\eta_0) \leq k(\sqrt{2})L_{\Omega}(\eta) = k(\sqrt{2})d_{\Omega}(\sigma(s), \sigma(t)).
\]

Consider now a **B-line** \( \sigma := ([a, a + ir] \cup [a + ir, a + 2r + ir] \cup [a + 2r + ir, a + 2r]) \cap \Omega \). If \( \sigma(s) \) and \( \sigma(t) \) are both either in \([a, a + ir] \) or in \([a + 2r + ir, a + 2r]\), it suffices to apply the previous argument. Hence, without loss of generality we can assume that \( \sigma(s) \in [a, a + ir] \) and \( \sigma(t) \in [a + 2r + ir, a + 2r] \), since the other cases are easier.

Let us consider a hyperbolic geodesic \( \eta \) joining \( \sigma(s) \) and \( \sigma(t) \). Denote by \( \eta_1 \) the subcurve of \( \eta \) starting at \( \sigma(s) \) with \( L_{\text{Eucl}}(\eta_1) = r - \text{Im}\ \sigma(s) \), and by \( \eta_2 \) the subcurve of \( \eta \) finishing in \( \sigma(t) \) with \( L_{\text{Eucl}}(\eta_2) = r - \text{Im}\ \sigma(t) \). Since the Euclidean length of \( \eta \) is at least \( 2r \), \( \eta_1 \) and \( \eta_2 \) are disjoint. Applying Lemma 3.5 twice we deduce

\[
 L_{\Omega}([\sigma(s), a + ir]) + L_{\Omega}([a + 2r + ir, \sigma(t)]) \leq k(\sqrt{2})L_{\Omega}(\eta_1) + k(\sqrt{2})L_{\Omega}(\eta_2) \\
 \leq k(\sqrt{2})L_{\Omega}(\eta) = k(\sqrt{2})d_{\Omega}(\sigma(s), \sigma(t)).
\]

We bound now \( L_{\Omega}([a + ir, a + 2r + ir]) \).

Let us consider a connected component \( \eta_* \) of \( \eta \cap \{ z \in \mathbb{C} : \text{Im}\ z \leq r \} \); then \( \eta_* \) joins \( z_1 := x_1 + iy_1 \) and \( z_2 := x_2 + iy_2 \), with \( 0 \leq y_1, y_2 \leq r \), and we define \( \sigma_* := [x_1 + ir, x_2 + ir] \).
Since $\Omega$ is a Denjoy domain, we conclude that $b \mapsto \lambda_\Omega(a+ib)$ is decreasing for $b > 0$ (see [26, Theorem 4.1(i)]); hence, $L_\Omega(\sigma_*) \leq L_\Omega(\eta_*)$.

Let us consider now the closure $\eta^*$ of a connected component of $\eta \cap \{z \in \mathbb{C} : \text{Im } z > r\}$; hence, $\eta^*$ joins $z_3 := x_3 + ir$ with $z_4 := x_4 + ir$, and we define $\sigma^* := [z_3, z_4]$. If $T := (z_4 - z_3)/2$, then we define $\sigma_1^* := [z_3, z_3 + T]$ and $\sigma_2^* := [z_3 + T, z_4]$.

Denote by $\eta_1^*$ the subcurve of $\eta^*$ finishing at $z_3$ with $L_{\text{Eucl}}(\eta_1^*) = T$, and by $\eta_2^*$ the subcurve of $\eta^*$ finishing at $z_4$ with $L_{\text{Eucl}}(\eta_2^*) = T$. Since the Euclidean length of $\eta^*$ is at least $2T$, $\eta_1^*$ and $\eta_2^*$ are disjoint. Since $\sigma$ is a $B$-line, we deduce that $\text{Im } z_3 = \text{Im } z_4 = r \geq T$. Therefore, applying Lemma 3.6 twice we deduce

$$L_\Omega(\sigma^*) = L_\Omega(\sigma_1^*) + L_\Omega(\sigma_2^*) \leq k(3)L_\Omega(\eta_1^*) + k(3)L_\Omega(\eta_2^*) \leq k(3)L_\Omega(\eta^*) .$$

Hence,

$$L_\Omega([a+ir, a+2r+ir]) \leq k(3)L_\Omega(\eta) ,$$

and consequently,

$$t - s = L_\Omega(\sigma|_{[s,t]}) \leq k(\sqrt{2})L_\Omega(\eta) + k(3)L_\Omega(\eta) = (k(\sqrt{2}) + k(3))d_\Omega(\sigma(s), \sigma(t)) .$$

This finishes the proof of (2).

Finally, let us consider a $C$-line $\sigma$. Applying a transformation $Tz = z + c$ if it is necessary, without loss of generality we can assume that the image of $\sigma$ is $\{x^2 + y^2 = r^2\}$ for some $r > 0$. Using part (2) of Lemma 3.9 we obtain

$$t - s = L_\Omega(\sigma|_{[s,t]}) \leq k(\sqrt{3})L_\Omega(F(\sigma)|_{[s,t]}) .$$

Since we have proved that $F(\sigma)$ is $k_1$-chord-arc, we have

$$L_\Omega(F(\sigma)|_{[s,t]}) \leq k_1d_\Omega(F(\sigma(t)), F(\sigma(s))) .$$

This inequality and part (4) of Lemma 3.9 give

$$L_\Omega(\sigma|_{[s,t]}) \leq k(\sqrt{3})k_1d_\Omega(F(\sigma(t)), F(\sigma(s))) \leq k(\sqrt{3})k_1d_\Omega(\sigma(t), \sigma(s)) .$$

This finishes the proof of the Theorem.

Using the same argument as in the proof of Theorem 4.2, and replacing always $k(c)$ by $c$ (see Remark 3.3), we obtain a similar result for the quasihyperbolic metric.

**Theorem 4.4.** Let $\Omega$ be any Denjoy domain. Then the following result holds for the quasihyperbolic metric:

1. Every $A$-line is $\sqrt{2}$-chord-arc.
2. Every $B$-line is $k'_1$-chord-arc, with $k'_1 := \sqrt{2} + 3$.
3. Every $C$-line is $k'_2$-chord-arc, with $k'_2 := 4\sqrt{3}k'_1$.

Now we prove that chord-arc curves are uniformly close to geodesics in every Denjoy domain.

**Theorem 4.5.** For every Denjoy domain $\Omega$ with its Poincaré metric, and for every $z, w \in \Omega \cap \mathbb{H}$, let $\gamma$ be the geodesic joining $z$ and $w$ in $\Omega \cap \mathbb{H}$ and let $g$ be the subarc of either an $A$-line, a $B$-line or a $C$-line joining $z$ and $w$. Then $H(\gamma, g) \leq M(\delta_0, k_2)$, where $\delta_0 := \log(1 + \sqrt{2})$, $k_2$ is the constant in Theorem 4.2, and $M(\delta, a)$ is the function in Theorem 2.6.
Remark 4.6. It can be proved that if we have either $z \in \Omega \cap \mathbb{H}$ or $w \in \Omega \cap \mathbb{H}$, then $\gamma$ is the unique geodesic joining $z$ and $w$ in $\Omega$. The same holds when $z$ and $w$ belong to the same connected component of $\Omega \cap \mathbb{R}$. If $z$ and $w$ belong to different connected components of $\Omega \cap \mathbb{R}$, then there are exactly two geodesics joining $z$ and $w$ in $\Omega$: $\gamma$ and its symmetric curve about the real axis.

Proof. Let us consider the bordered Riemann surface $\Omega^+ = \Omega \cap \mathbb{H}$. By [4, p.130], we know that the unit disk and the upper halfplane are $\delta_0$-hyperbolic. Since $\Omega$ is symmetric about the real axis, we have that the Poincaré metric in $\Omega$ is also symmetric about the real axis, i.e. $\lambda_\Omega(z) = \lambda_\Omega(z)$ for every $z \in \Omega$; this implies that each connected component of $\Omega \cap \mathbb{R}$ is a geodesic. $\Omega^+$ is isometric to a geodesically convex subset of the unit disk, since it is a simply connected set bounded by disjoint geodesics; therefore, it is $\delta_0$-hyperbolic also.

By Theorem 4.2, $g$ is $k_2$-chord-arc, with $k_2$ the constant in Theorem 4.2.

By Theorem 2.6, we have $g \subset \mathcal{N}_A(\gamma)$, with $A = M(\delta_0, k_2)$ and $M(\delta, a)$ the function in Theorem 2.6. Hence, we just need to show that given any point $p \in \gamma$, we have $d_\Omega(p, g) \leq A$.

Let $\pi: \mathbb{D} \rightarrow \Omega$ be a universal covering map and let $\tilde{\gamma}$ and $\tilde{\gamma}$, respectively, be lifts of $\Omega^+$ and $\gamma$ in $\mathbb{D}$ with $\tilde{\gamma} \subseteq \Omega^+$. Since the set $\mathcal{N}_A(\gamma)$ is geodesically convex in $\mathbb{D}$, then $\mathcal{N}_A(\gamma) \cap \Omega^+$ is geodesically convex in $\mathbb{D}$, and therefore it is simply connected. Since $\pi: \tilde{\Omega}^+ \rightarrow \Omega^+$ is a biholomorphic map, the set $E := \pi(\mathcal{N}_A(\gamma) \cap \Omega^+)$ is simply connected. Note that $E$ is the connected component of $\mathcal{N}_A(\gamma) \cap \Omega^+$ containing $\gamma$. Given any point $p \in \gamma$, we denote by $\eta_p$ the local geodesic orthogonal to $\gamma$ from $p$. Since $\Omega^+$ is isometric to a geodesically convex subset of the unit disk, the connected component $\eta_p^*$ of $\eta_p \cap \Omega^+$ containing $p$ is a geodesic; therefore, $d_\Omega(z_0, g) = d_\Omega(z_0, q)$ for every $z_0 \in \eta_p^*$. Since $E$ is a simply connected set, $E \setminus \eta_p^*$ is not connected. The points $z$ and $w$ are in different connected components of $E$; then, since $g$ is connected, there is a point $q \in g \cap \eta_p^*$, and consequently, $d_\Omega(p, g) \leq d_\Omega(p, q) \leq A$. \qed

We also have a similar result for the quasihyperbolic metric, without a beautiful expression for the constant.

Theorem 4.7. For every Denjoy domain $\Omega$ with its quasihyperbolic metric, and for every $z, w \in \Omega \cap \mathbb{H}$, let $\gamma$ be a geodesic joining $z$ and $w$ in $\Omega \cap \mathbb{H}$ and let $g$ be the subarc of either an $A$-line, a $B$-line or a $C$-line joining $z$ and $w$. Then $H(\gamma, g) \leq H_0$, for some universal constant $H_0$.

Proof. Let us consider the bordered Riemann surface $\Omega^+ = \Omega \cap \mathbb{H}$. This set $\Omega^+$ with its quasihyperbolic metric is $c$-hyperbolic for a universal constant $c$ (see [20, Lemma 3.1]).

By Theorem 4.4, $g$ is $k_2'$-chord-arc (is a $(k'_2, 0)$-quasigeodesic), with $k'_2 := 4 \sqrt{3} (\sqrt{2} + 3)$.

By Theorem 2.5, we have $H(\gamma, g) \leq H_0$, for some universal constant $H_0$ (it just depends on $c$ and $k'_2$, which are universal constants). \qed

5. DISTANCE ESTIMATES AND LENGTHS OF SIMPLE CLOSE GEODESICS

Using the results in the previous sections we obtain here good estimates for the Poincaré distance of:

(i) any couple of points $z, w \in \Omega \cap \mathbb{R}$ (see Theorem 5.3),

(ii) any couple of connected components of $\Omega \cap \mathbb{R}$ (see Theorem 5.4),

(iii) any point $z \in \Omega \cap \mathbb{R}$ and any connected component of $\Omega \cap \mathbb{R}$ (see Theorem 5.6).
In this section we just consider the Poincaré metric, since there exists a simple function comparable to the quasihyperbolic distance for every Denjoy domain (see e.g. [21, Lemma 5.1]), which allows to solve these three problems for this latter metric.

We obtain these estimates up to multiplicative constants, which are the best possible results for the Poincaré metric, since the sharpest known estimates for the density of the Poincaré metric in Theorem 2.8 also have this property.

Note that (ii) is equivalent to estimate the length of simple closed geodesics, a very interesting and difficult problem for the Poincaré metric. These geodesics are a main object in Riemannian geometry. The closed geodesics are the periodic orbits of the dynamical system associated to a manifold on its unit tangent bundle, and they provide tools to study the geodesic flow, just like the fixed points of an automorphism helps to study it. Lastly, the closed geodesics are becoming more and more important in the study of heat and wave equations, and of the spectrum of the manifold. The lengths of all closed geodesics determine largely the spectrum. Conversely, the spectrum determines completely the lengths of the closed geodesics (see [11], [18], [13]).

**Lemma 5.1.** Let $\Omega$ be any Denjoy domain, $a \in \mathbb{R}$ and $r > 0$. Then we have

\[
L_\Omega([a + ir, a \pm r + ir]) \leq k(2) L_\Omega([a, a + ir]) .
\]

**Proof.** We are going to prove that

\[
\lambda_\Omega(a + t + ir) \leq k(2) \lambda_\Omega(a + ir)
\]

for every real $t$ with $|t| \leq r$. Since $\Omega$ is a Denjoy domain, we conclude that $b \mapsto \lambda_\Omega(a + ib)$ is decreasing for $b > 0$ (see [26, Theorem 4.1(i)]), and then $\lambda_\Omega(a + t + ir) \leq k(2) \lambda_\Omega(a + i(r - |t|))$ for every $t \in [0, r]$. This inequality proves the Lemma, since the three intervals involved have the same Euclidean length.

Let us prove now (5.2). Choose $a_t \in \partial \Omega$ with $\delta_\Omega(a + t + ir) = |a + t + ir - a_t|$. Let us note that

\[
|a + ir - a_t| \leq |a + ir - (a + t + ir)| + |a + t + ir - a_t| = |t| + |a + t + ir - a_t| \leq r + |a + t + ir - a_t| \leq 2|a + t + ir - a_t| .
\]

Therefore, Lemma 3.2 gives $\lambda_\Omega(a + t + ir) \leq k(2) \lambda_\Omega(a + ir)$. \hfill $\square$

The next result allows to estimate the distance of any couple of points of $\Omega \cap \mathbb{R}$ in $\Omega$.

**Theorem 5.3.** Let $\Omega$ be any Denjoy domain and $g$ any $B$-line. Then we have

\[
\frac{1}{k(2) + 1} L_\Omega(g) \leq L_\Omega([a, a + ir] \cup [a + 2r, a + 2r + ir]) < L_\Omega(g) .
\]

Furthermore,

\[
\frac{1}{k(2) + 1} d_\Omega(a, a + 2r) \leq L_\Omega([a, a + ir] \cup [a + 2r, a + 2r + ir]) < k_1 d_\Omega(a, a + 2r) ,
\]

for every $a, a + 2r \in \mathbb{R}$, with $k_1 = k(\sqrt{2}) + k(3)$.

**Proof.** Applying Lemma 5.1 twice we obtain for every $B$-line $g$

\[
L_\Omega([a + ir, a + 2r + ir]) \leq k(2) L_\Omega([a, a + ir] \cup [a + 2r, a + 2r + ir]) ,
\]

\[
L_\Omega(g) = L_\Omega([a, a + ir]) + L_\Omega([a + ir, a + 2r + ir]) + L_\Omega([a + 2r, a + 2r + ir]) \leq (k(2) + 1) L_\Omega([a, a + ir] \cup [a + 2r, a + 2r + ir]) ,
\]

\[
\frac{1}{k(2) + 1} d_\Omega(a, a + 2r) \leq L_\Omega([a, a + ir] \cup [a + 2r, a + 2r + ir]) < k_1 d_\Omega(a, a + 2r) ,
\]

for every $a, a + 2r \in \mathbb{R}$, with $k_1 = k(\sqrt{2}) + k(3)$. 

which is the first inequality in the second display. The first one is trivial.

In order to finish the proof we just need to note that  
\[ d_Ω(a, a + 2r) \leq L_Ω(g) \leq k_1 d_Ω(a, a + 2r), \]  
by Theorem 4.2.

The next result allows to estimate the distance of any couple of connected components of  \( Ω \cap \mathbb{R} \) or, equivalently, the length of simple closed geodesics in  \( Ω \).

**Theorem 5.4.** Let  \( Ω \) be any Denjoy domain with  \( Ω \cap \mathbb{R} = \cup_n (a_n, b_n) \). Denote by  \( x_n \) the midpoint of  \( (a_n, b_n) \) and by  \( γ_{mn} \) the shorter geodesic joining  \( (a_m, b_m) \) and  \( (a_n, b_n) \), with  \( a_m < a_n \). There exist universal constants  \( c_1, c_2 \) and  \( c_3 \), verifying the following:

1. If  \( b_m - a_m \leq a_n - b_m \) and  \( b_n - a_n \leq a_m - b_m \), then  
   \[ c_1 L_Ω(γ_{mn}) \leq L_Ω([x_m, x_m + i|x_n - x_m|/2]) + L_Ω([x_n, x_n + i|x_n - x_m|/2]) \leq c_2 L_Ω(γ_{mn}). \]

2. If  \( b_m - a_m \leq b_n - a_m, a_n - b_m \leq b_n - a_m, \) and  
   \[ r(a_m, b_m, a_n, b_n) := \frac{(b_m - a_m)(b_n - a_n)}{(a_n - b_m)(b_n - a_m)} \leq r_0, \]
   for some positive constant  \( r_0 \), then  
   \[ c_3 L_Ω(γ_{mn}) \leq L_Ω([x_n, x_n + i(x_n - b_m)]) \leq c_2 (3r_0 + 2) L_Ω(γ_{mn}). \]

In fact, we can choose  
\[ c_1 = \frac{1}{k(2)+1}, \quad c_2 = 2k(1)(k(\sqrt{2}) + k(3)), \quad c_3 = \frac{1}{(k(2)+1)(k(3\sqrt{2}) + 1)}. \]

3. If  \( r(a_m, b_m, a_n, b_n) \geq r_0 \) for some  \( r_0 > 1 \), then there exist constants  \( c_4, c_5 \), which just depend on  \( r_0 \), such that  
   \[ c_4 L_Ω(γ_{mn}) \leq \frac{1}{\log r(a_m, b_m, a_n, b_n)} \leq c_5 L_Ω(γ_{mn}). \]

**Remark 5.5.** 1. By symmetry, we can assume always  \( a_m < a_n \) and  \( b_n - a_n \leq b_m - a_m \); therefore, these hypotheses are just technical, and Theorem 5.4 covers all possible cases.

2. We also allow  \( a_m = -\infty \). The case  \( a_m = -\infty \) and  \( b_n = \infty \) is direct, since then  \( \infty \) is a puncture and  \( L_Ω(γ_{mn}) = 0 \).

3. Although  \( b_n - a_m > 0 \), it is possible to have  \( a_n - b_m = 0 \), and then  \( r(a_m, b_m, a_n, b_n) = \infty \) (therefore  \( a_n = b_m \) and  \( L_Ω(γ_{mn}) = 0 \)).

**Proof.** Recall that the first part of Theorem 5.3 states that for every  \( B \)-line  \( g \)  
\[ L_Ω(g) \leq (k(2)+1) L_Ω([a, a + ir] \cup [a + 2r + ir, a + 2r]). \]

Note that, since the map  \( b \mapsto \lambda_Ω(a + ib) \) is decreasing for  \( b > 0 \) (see [26, Theorem 4.1(i)]), we have for every constant  \( Q \geq 1 \)  
\[ L_Ω([x, x + iQy]) \leq Q L_Ω([x, x + iy]). \]

If  \( η := [x, x + iy] \), we denote by  \( Q_η \) the segment  \( Q_η := [x, x + iQy] \); then  \( L_Ω(Q_η) \leq Q L_Ω(η) \).

Let us consider the  \( B \)-line  \( B \) joining  \( x_m \) and  \( x_n \). If  \( y_m \) and  \( y_n \) are the endpoints of  \( γ_{mn} \), consider the  \( B \)-line  \( B' \) joining  \( y_m \) and  \( y_n \).

Let us denote by  \( σ_j \) (respectively,  \( σ'_j \)) the vertical segment of  \( B \) (respectively,  \( B' \)) starting in  \( (a_j, b_j) \), for  \( j = m, n \). We define  \( σ_n := [x_n, x_n + i(x_n - b_m)] \).

We denote by  \( h \) (respectively,  \( h', h'' \)) the maximum of the imaginary part of the points in  \( σ_n \) (respectively,  \( σ'_n, σ_n \)).
We prove first (1). Then \(b_m - a_m \leq a_n - b_m\) and \(b_n - a_n \leq a_n - b_m\) imply
\[
2h' = y_n - y_m \geq a_n - b_m \geq \frac{x_n - x_m}{2} = h.
\]

If \(\zeta\) is a point in \(\{a_j, b_j\} (j = m, n)\) with \(\delta \Omega(y_j) = |y_j - \zeta|\), then we also have \(\delta \Omega(x_j) = |x_j - \zeta|\), since \(|x_j - a_j| = |x_j - b_j|\). Hence, \(|y_j + it - \zeta| \leq |x_j + it - \zeta|\). Since \(h \leq 2h'\), we have \(\sigma_j \subseteq 2\sigma_j'\) and Lemma 3.4 gives \(L_\Omega(\sigma_j) \leq k(1) L_\Omega(2\sigma_j') \leq 2 k(1) L_\Omega(\sigma_j')\). Therefore, using Theorem 4.2
\[
\frac{1}{k(2) + 1} L_\Omega(\gamma_{mn}) \leq \frac{1}{k(2) + 1} L_\Omega(B) \leq L_\Omega(\sigma_m) + L_\Omega(\sigma_n)
\]
\[
\leq 2 k(1) L_\Omega(\sigma_m') + 2 k(1) L_\Omega(\sigma_n')
\]
\[
\leq 2 k(1) L_\Omega(B') \leq 2 k(1) (k(\sqrt{2}) + k(3)) L_\Omega(\gamma_{mn}).
\]

We prove now (2). Since \(b_n - a_n \leq b_n - a_m\) and \(a_n - b_n \leq b_m - a_n\), we have \(b_n - a_m \leq 3(b_n - a_m)\) and, consequently, \(b_n - a_n \leq 3r_0(a_n - b_m)\).

We distinguish two cases.

(a) We assume first that \((b_n - a_m)/2 \leq x_n - b_m\). We have
\[
x_n - b_m = \frac{b_n - a_n}{2} + a_n - b_m \leq \frac{b_n - a_m}{2} + b_m - a_m = 3(b_m - x_m),
\]
and hence,
\[
|x_n + it - b_m| \leq t + x_n - b_m \leq 3\sqrt{2} \frac{1}{\sqrt{2}} (t + b_m - x_m) \leq 3\sqrt{2}|x_n + it - b_m| = 3\sqrt{2} \delta \Omega(x_n + it).
\]

Lemma 3.4 gives \(L_\Omega(\sigma_m) \leq k(3\sqrt{2}) L_\Omega(\sigma_n).\) Therefore,
\[
\frac{1}{(k(2) + 1)(k(\sqrt{2}) + 1)} L_\Omega(\gamma_{mn}) \leq \frac{1}{(k(2) + 1)(k(\sqrt{2}) + 1)} L_\Omega(B)
\]
\[
\leq \frac{1}{k(\sqrt{2}) + 1} (L_\Omega(\sigma_m) + L_\Omega(\sigma_n)) \leq L_\Omega(\sigma_n).
\]

Since we are assuming \((b_n - a_m)/2 \leq x_n - b_m\), we have
\[
h = \frac{x_n - x_m}{2} = \frac{x_n - b_m}{2} + \frac{b_m - x_m}{2} = \frac{b_n - b_m}{2} + \frac{b_m - a_m}{4} \leq \frac{x_n - b_m}{2} + \frac{x_n - b_m}{2} = \hat{h},
\]
and then \(h \leq \hat{h} \leq 2h\). Therefore, \(\sigma_n \subseteq \sigma_n' \subseteq 2\sigma_n\) and
\[
L_\Omega(\sigma_n) \leq L_\Omega(\sigma_n') \leq L_\Omega(2\sigma_n) \leq 2 L_\Omega(\sigma_n).
\]

We also have
\[
h \leq \hat{h} = x_n - b_m = x_n - a_n + a_n - b_m = \frac{1}{2}(b_n - a_n + 2(a_n - b_m))
\]
\[
\leq \frac{1}{2}(3r_0(a_n - b_m) + 2(a_n - b_m)) \leq (3r_0 + 2)h',
\]
and then $h \leq 2h \leq 2(3r_0 + 2)h'$. A similar argument to the one in the proof of (1), using Lemma 3.4, gives $L_\Omega(\bar{\sigma}_m) \leq k(1) L_\Omega(2(3r_0 + 2)\sigma'_n) \leq 2(3r_0 + 2)k(1) L_\Omega(\sigma'_n)$. Hence, using Theorem 4.2,

$$
\frac{1}{(k(2) + 1)(k(3\sqrt{2}) + 1)} L_\Omega(\gamma_{mn}) \leq L_\Omega(\sigma_n) \leq L_\Omega(\bar{\sigma}_n) \leq 2(3r_0 + 2)k(1) L_\Omega(\sigma'_n)
$$

\leq 2(3r_0 + 2)k(1) L_\Omega(B')

\leq 2(3r_0 + 2)k(1) k_1 L_\Omega(\gamma_{mn})

= c_2 (3r_0 + 2) L_\Omega(\gamma_{mn}).

(b) We consider now the case $x_n - b_m < (b_m - a_m)/2$.

Note that in this case it is possible that $x_m$ is not well defined, since the case $a_m = -\infty$ is allowed, and then $x_m = -\infty$. We define $B$ in this case as the $B$-line joining $x^*_m := 2b_m - x_n$ and $x_n$. Note that, by our hypothesis,

$$
x^*_m = 2b_m - x_n = b_m - (x_n - b_m) > b_m - \frac{b_m - a_m}{2} = x_m,
$$

and then $x^*_m$ is nearest from $b_m$ than $x_m$; hence, $\delta(x^*_m + it) = |x^*_m + it - b_m|$. We also have $h = h = x_n - b_m$ and $\bar{\sigma}_n = \sigma_n$. Then

$$
h = x_n - b_m = \frac{b_m - a_m}{2} + 2\frac{a_n - b_m}{2} \leq (3r_0 + 2)\frac{a_n - b_m}{2} \leq (3r_0 + 2)h'.
$$

A similar argument to the one in the proof of (1), using Lemma 3.4, gives $L_\Omega(\sigma_n) \leq k(1) L_\Omega((3r_0 + 2)\sigma'_n) \leq (3r_0 + 2)k(1) L_\Omega(\sigma'_n)$.

We also have

$$
|x_n + it - b_m| = |x^*_m + it - b_m| = \delta_\Omega(x^*_m + it),
$$

and then Lemma 3.4 gives $L_\Omega(\sigma_m) \leq k(1) L_\Omega(\sigma_n)$. Therefore, using Theorem 4.2,

$$
\frac{1}{(k(2) + 1)(k(1) + 1)} L_\Omega(\gamma_{mn}) \leq \frac{1}{(k(2) + 1)(k(1) + 1)} L_\Omega(B)
$$

\leq \frac{1}{k(1) + 1} (L_\Omega(\sigma_m) + L_\Omega(\sigma_n)) \leq L_\Omega(\sigma_n)

\leq (3r_0 + 2)k(1) L_\Omega(\sigma'_n) \leq (3r_0 + 2)k(1) L_\Omega(B')

\leq (3r_0 + 2)k(1) k_1 L_\Omega(\gamma_{mn})

\leq c_2 (3r_0 + 2) L_\Omega(\gamma_{mn}).

This finishes the proof of (2).

Finally, we prove (3). Assume that $r : = r(a_m, b_m, a_n, b_n) \geq r_0$ for some $r_0 > 1$. Let us consider the Möbius map

$$
T(z) : = \frac{(b_m - a_m)(z - a_n)}{(a_n - b_m)(z - a_m)}.
$$

It is clear that $T(a_m) = \infty$, $T(b_m) = -1$, $T(a_n) = 0$ and $T(b_n) = r$; if we define $S_r : = \mathbb{C} \setminus \{-1, 0, r\}$ and $T_r : = \mathbb{C} \setminus \{[-1, 0] \cup [r, \infty]\}$, then $T_r \subset T(\Omega) \subset S_r$. It is easy to check that $\sigma_r := \{z \in \mathbb{C} : |z + 1| = \sqrt{1 + r}\}$ is the simple closed geodesic in $S_r$ (and in $T_r$) which surrounds $\{-1, 0\}$ and does not surround $\{r\}$. Since $T_r \subset T(\Omega) \subset S_r$, we have

$$
L_{S_r}(\sigma_r) \leq L_{T(\Omega)}(T(\gamma_{mn})) = L_\Omega(\gamma_{mn}) \leq L_{T_r}(\sigma_r).
$$
Then, we just need to apply [2, Lemma 4.5]. This finishes the proof.

The next result allows to estimate the distance from any connected component of \( \Omega \cap \mathbb{R} \) to a point of \( \Omega \cap \mathbb{R} \).

**Theorem 5.6.** Let \( \Omega \) be any Denjoy domain with \( \Omega \cap \mathbb{R} = \bigcup_n (a_n, b_n) \). Given \( x \in (a_n, b_n) \), denote by \( \gamma^x_m \) the shorter geodesic joining \( x \) and \( (a_m, b_m) \). There exist universal constants \( c_1, c_2, C_1 \) and \( C_2 \), verifying the following:

1. If \( b_m - a_m \leq 2d_{\text{Euc}}(x, (a_m, b_m)) \), then
   \[
   c_1 L_\Omega(\gamma^x_m) \leq L_\Omega([x_m, x_m + i|x - x_m|/2]) + L_\Omega([x, x + i|x - x_m|/2]) \leq c_2 L_\Omega(\gamma^x_m).
   \]
   where \( x_m \) is the midpoint of \( (a_m, b_m) \).

2. If \( b_m - a_m > 2d_{\text{Euc}}(x, (a_m, b_m)) \), then
   \[
   C_1 L_\Omega(\gamma^x_m) \leq L_\Omega([x, x + i(x - b_m)]) \leq C_2 L_\Omega(\gamma^x_m).
   \]

In fact, we can choose \( c_1, c_2 \) as in Theorem 5.4,

\[
C_1 = \frac{1}{(k(2) + 1)(k(1) + 1)}, \quad C_2 = 2(k(\sqrt{2}) + k(3)).
\]

**Proof.** By symmetry, without loss of generality we can assume that \( a_m < a_n \).

Recall that the first part of Theorem 5.3 states that for every \( B \)-line \( g \)

\[
L_\Omega(g) \leq (k(2) + 1) L_\Omega([a, a + ir] \cup [a + 2r, a + 2r + ir]).
\]

As we have seen in the beginning of the proof of Theorem 5.4, we also have

\[
L_\Omega([x_0, x_0 + iQy_0]) \leq QL_\Omega([x_0, x_0 + iy_0]) \quad \text{for every} \quad x_0 \in \mathbb{R}, \quad y_0 > 0 \quad \text{and} \quad Q \geq 1.
\]

If \( b_m - a_m < 2(x - b_m) \), we have defined \( x_m \) as the midpoint of \( (a_m, b_m) \). If \( b_m - a_m > 2(x - b_m) \), let us define \( x_m \) as \( x_m := 2b_m - x \).

Let us consider the \( B \)-line \( B \) joining \( x_m \) and \( x \). If \( y_m := \gamma^x_m \cap (a_m, b_m) \), consider the \( B \)-line \( B' \) joining \( y_m \) and \( x \).

Let us denote by \( \sigma_j \) (respectively, \( \sigma'_j \)) the vertical segment of \( B \) (respectively, \( B' \)) starting in \((a_j, b_j)\), for \( j = m, n \).

We denote by \( h \) (respectively, \( h' \)) the maximum of the imaginary part of the points in \( \sigma_n \) (respectively, \( \sigma'_n \)).

We prove first (1). Then \( b_m - a_m \leq 2(x - b_m) \) and this implies

\[
2h' = x - y_m \geq x - b_m \geq \frac{1}{2} (x - b_m + \frac{b_m - a_m}{2}) = h.
\]

If \( \zeta \) is a point in \( \{a_m, b_n\} \) with \( \delta_\Omega(y_m) = |y_m - \zeta| \), then we also have \( \delta_\Omega(x_m) = |x_m - \zeta| \), since \( |x_m - a_m| = |x_m - b_m| \). Hence, \( |y_m + it - \zeta| \leq |x_m + it - \zeta| \). Since \( h' \leq 2h' \), we have \( \sigma_j \subseteq 2\sigma'_j \); then \( L_\Omega(\sigma_m) \leq L_\Omega(2\sigma'_m) \leq 2L_\Omega(\sigma'_m) \) and Lemma 3.4 gives \( L_\Omega(\sigma_m) \leq k(1) L_\Omega(2\sigma'_m) \leq 2k(1) L_\Omega(\sigma'_m) \). Therefore, using Theorem 4.2

\[
\frac{1}{k(2) + 1} L_\Omega(\gamma^x_m) \leq \frac{1}{k(2) + 1} L_\Omega(B) \leq L_\Omega(\sigma_m) + L_\Omega(\sigma_n) \leq 2k(1) L_\Omega(\sigma'_m) + 2L_\Omega(\sigma'_n) \leq 2k(1) L_\Omega(B') \leq 2k(1) (k(\sqrt{2}) + k(3)) L_\Omega(\gamma^x_m).
\]
We prove now (2). Since \(2(x - b_m) < b_m - a_m\) and \(x_m = 2b_m - x\), we have \(x_m > (a_m + b_m)/2\), \(b_m - x_m = \delta \Omega (x_m)\) and \(x - b_m = b_m - x_m\). Hence, \(|x - b_m + it| = |x_m + it - b_m| = \delta \Omega (x_m + it)\). Lemma 3.4 gives \(L_\Omega (\sigma_m) \leq k(1) L_\Omega (\sigma_n)\). Therefore,

\[
\frac{1}{(k(2)+1)(k(1)+1)} L_\Omega (\gamma^2_m) \leq \frac{1}{(k(2)+1)(k(1)+1)} L_\Omega (B) \\
\leq \frac{1}{k(1)+1} \left( L_\Omega (\sigma_m) + L_\Omega (\sigma_n) \right) \leq L_\Omega (\sigma_n) .
\]

Note that \(2h' = x - y_m \geq x - b_m = h\). Therefore, \(\sigma_n \leq 2\sigma'_n\) and using Theorem 4.2

\[
L_\Omega (\sigma_n) \leq L_\Omega (2\sigma'_n) \leq 2 L_\Omega (\sigma'_n) \leq 2 L_\Omega (B') \leq 2 \left(k(\sqrt{2}) + k(3)\right) L_\Omega (\gamma^2_m) .
\]

\[\Box\]

We need a last technical Lemma.

**Lemma 5.7.** Let us consider a Denjoy domain \(\Omega\), \(x \in \Omega \cap \mathbb{R}\) and \(0 \leq u < v\). Then

\[
\frac{2^{-3/2}(v-u)}{k(\sqrt{1+u^2}/\sqrt{1+u^2})} \sqrt{1 + u^2} (k_0 + \beta \Omega (x + iu\delta \Omega (x))) \leq L_\Omega ([x + iu\delta \Omega (x), x + iv\delta \Omega (x)]) \leq \frac{(v-u)k(1)\pi}{4 \sqrt{1 + u^2} (k_0 + \beta \Omega (x + iu\delta \Omega (x)))} .
\]

**Proof.** Let us consider \(0 \leq u \leq y \leq v\), and \(a \in \partial \Omega\) with \(\delta \Omega (x) = |x - a|\). We obviously have \(\delta \Omega (x + iy\delta \Omega (x)) = |x + iy\delta \Omega (x) - a|\). It is easy to check that

\[
1 \leq \frac{|x - iy\delta \Omega (x) - a|}{|x - iy\delta \Omega (x) - a|} = \frac{\delta \Omega (x)\sqrt{1 + y^2}}{\delta \Omega (x)\sqrt{1 + u^2}} \leq \frac{\sqrt{1 + y^2}}{\sqrt{1 + u^2}} ,
\]

and applying Lemma 3.2 we obtain

\[
\frac{1}{k(1)} \lambda \Omega (x + iu\delta \Omega (x)) \leq \lambda \Omega (x + iu\delta \Omega (x)) \leq k \left(\frac{\sqrt{1 + y^2}}{\sqrt{1 + u^2}}\right) \lambda \Omega (x + iu\delta \Omega (x)) .
\]

Consequently, using Theorem 2.8,

\[
L_\Omega ([x + iu\delta \Omega (x), x + iv\delta \Omega (x)]) = \int_u^v \lambda \Omega (x + iy\delta \Omega (x)) \delta \Omega (x) dy \\
\leq (v-u)\delta \Omega (x)k(1)\lambda \Omega (x + iu\delta \Omega (x)) \\
\leq \frac{(v-u)k(1)\pi}{4 \sqrt{1 + u^2} (k_0 + \beta \Omega (x + iu\delta \Omega (x)))} ,
\]

\[
L_\Omega ([x + iu\delta \Omega (x), x + iv\delta \Omega (x)]) \geq \frac{(v-u)\delta \Omega (x)}{k(\sqrt{1+u^2}/\sqrt{1+v^2})} \lambda \Omega (x + iu\delta \Omega (x)) \\
\geq \frac{2^{-3/2}(v-u)}{k(\sqrt{1+u^2}/\sqrt{1+v^2})} \sqrt{1 + u^2} (k_0 + \beta \Omega (x + iu\delta \Omega (x))) .
\]

\[\Box\]
Theorems 5.3, 5.4 and 5.6 estimate distances (not easy at all to compute) in terms of lengths of vertical segments. The following result gives a practical criterion in order to estimate $L_{\Omega}([a, a + ir])$ in a simple way, by using a comparable quantity (easy to compute).

We define, as usual, the integer part of $x \in \mathbb{R}$ as $[x] := n$ if $x \in [n, n + 1)$.

**Theorem 5.8.** Let us consider a Denjoy domain $\Omega$, $a \in \Omega \cap \mathbb{R}$, $r > 0$ and $m := \lfloor \log_2(r/\delta_{\Omega}(a)) \rfloor$. Then:

1. If $r \geq \delta_{\Omega}(a)$ ($m \geq 0$),
   \[
   \frac{1}{4k(2)} \left( \frac{1}{k_0 + \beta_{\Omega}(a)} + \sum_{n=0}^{m-1} \frac{1}{k_0 + \beta_{\Omega}(a + i2^n\delta_{\Omega}(a))} \right) \leq L_{\Omega}([a, a + ir]) \leq \frac{k(1)\pi}{2} \left( \frac{1}{k_0 + \beta_{\Omega}(a)} + \sum_{n=0}^{m-1} \frac{1}{k_0 + \beta_{\Omega}(a + i2^n\delta_{\Omega}(a))} \right).
   \]

2. If $r < \delta_{\Omega}(a)$,
   \[
   \frac{2^{-3/2}}{k(\sqrt{2})} \cdot \frac{r}{\delta_{\Omega}(a)(k_0 + \beta_{\Omega}(a))} \leq L_{\Omega}([a, a + ir]) \leq \frac{k(1)\pi}{4} \cdot \frac{r}{\delta_{\Omega}(a)(k_0 + \beta_{\Omega}(a))}.
   \]

**Remark 5.9.** As usual, we define $\sum_{n=0}^{-1} := 0$.

**Proof.** In order to prove (1), note that $m \leq \log_2(r/\delta_{\Omega}(a)) < m + 1$, and therefore $2^m \delta_{\Omega}(a) \leq r < 2^{m+1} \delta_{\Omega}(a)$. Recall that, as we have seen in the beginning of the proof of Theorem 5.4, we have $L_{\Omega}(Q\eta) \leq Q L_{\Omega}(\eta)$ for every constant $Q \geq 1$. Therefore,

$L_{\Omega}([a, a + i2^m\delta_{\Omega}(a)]) \leq L_{\Omega}([a, a + ir]) \leq L_{\Omega}([a, a + i2^{m+1}\delta_{\Omega}(a)]) \leq 2 L_{\Omega}([a, a + i2^m\delta_{\Omega}(a)])$,

and

$L_{\Omega}([a, a + i2^m\delta_{\Omega}(a)]) = L_{\Omega}([a, a + i\delta_{\Omega}(a)]) + \sum_{n=0}^{m-1} L_{\Omega}([a + i2^n\delta_{\Omega}(a), a + i2^{n+1}\delta_{\Omega}(a)])$.

If $u = 0$ and $v = 1$, then

\[
\frac{v - u}{\sqrt{1 + u^2}} = 1, \quad \frac{\sqrt{1 + v^2}}{\sqrt{1 + u^2}} = \sqrt{2};
\]

if $u = 2^n$ and $v = 2^{n+1}$, then

\[
\frac{v - u}{\sqrt{1 + u^2}} = \frac{2^n}{\sqrt{1 + 2^{2n}}} \in [1/\sqrt{2}, 1], \quad \frac{\sqrt{1 + v^2}}{\sqrt{1 + u^2}} = \frac{\sqrt{1 + 2^{2n+2}}}{\sqrt{1 + 2^{2n}}} < 2.
\]

These facts and Lemma 5.7 give (1).

Let us prove now (2). If $u = 0$ and $v = r/\delta_{\Omega}(a)$, then

\[
\frac{v - u}{\sqrt{1 + u^2}} = \frac{r}{\delta_{\Omega}(a)}, \quad \frac{\sqrt{1 + v^2}}{\sqrt{1 + u^2}} < \sqrt{2}.
\]

These facts and Lemma 5.7 give (2). \qed
6. Isoperimetric inequalities

Let us consider a non-exceptional Riemann surface $S$ with its Poincaré metric. We say that $S$ satisfies the linear isoperimetric inequality (LII) if there exists a constant $h > 0$ such that for every relatively compact domain (open and connected set) $G$ with smooth boundary one has that

\[
A_S(G) \leq h L_S(\partial G).
\]

We denote by $h(S)$ the best constant in (6.1).

There are a number of natural questions concerning the LII-property of Riemann surfaces. Particularly interesting are the stability under appropriate maps, its relation with other conformal invariants and its characterization for plane domains.

Concerning the study of the stability of LII, in [14, Theorem 1] it was proved that LII is invariant by quasiconformal maps.

One of the conformal invariants related with the LII-property is the bottom of the spectrum of the Laplace-Beltrami operator, $b(S)$, defined in terms of Rayleigh’s quotient. The number $b(S)$ belongs to $[0, 1/4]$ and a celebrated theorem of Elstrodt, Patterson and Sullivan [38, p. 333] relates it with other important conformal invariant of $S$, its exponent of convergence $\delta(S)$ (see e.g. [27, p. 21] for basic background). It is a well known fact that $0 \leq \delta(S) \leq 1$ (see e.g. [27, p. 21]).

It is also well known (see e.g., [10, p. 95], [12], [14, Theorem 2]) that

\[
\frac{1}{4} \leq b(S) h(S)^2 \quad \text{and} \quad b(S) h(S) \leq \frac{3}{2}.
\]

Therefore $S$ has the LII-property if and only if $b(S) > 0$ or, equivalently, $\delta(S) < 1$.

It is also known that $\delta(S)$ coincides with the Hausdorff dimension of the conical limit set of the covering group of $S$ (see e.g. [27, p. 154]). This says us that the LII-property must be also related with the size of the “boundary” of $S$.

Although the characterization of LII for plane domains is a very difficult problem, there exists such a characterization of LII for Denjoy domains in [2]. We need some definitions in order to explain this result.

**Definition 6.2.** A subset $I$ of a non-exceptional Riemann surface $S$ is strongly uniformly separated in $S$, if there exists a positive constant $r_0$ such that the hyperbolic balls $B_S(p, r_0)$, where $p \in I$, are simply connected and pairwise disjoint.

**Definition 6.3.** Given a Denjoy domain $\Omega$ we denote by $I = I(\Omega)$ the isolated points of $\partial \Omega$, and we define $\Omega_0 := \Omega \cup I$. Then $\Omega_0$ is also a Denjoy domain and $\Omega = \Omega_0 \setminus I$.

**Definition 6.4.** We say that a finite subset $A = \{\alpha_1, \ldots, \alpha_{2n}\}$ ($n \geq 2$) of points of $\partial \Omega \cup \{\infty\}$ is a border set of $\partial \Omega$ if $A$ verifies the following two conditions:

(i) $A$ is “ordered” in $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, i.e. there exists $j \in \mathbb{Z}_{2n} = \mathbb{Z}/(2n\mathbb{Z})$ such that $\alpha_{j+1} < \cdots < \alpha_{j+2n}$, where the subscripts belong to $\mathbb{Z}_{2n}$.

(ii) The set $\bigcup_{k=1}^{n}(a_{2k-1}, a_{2k})$ is contained in $\Omega$.

Obviously every subset $A = \{\alpha_1, \ldots, \alpha_{2n}\}$ of $\bar{\mathbb{R}}$ can be “ordered” in such a way that the condition (i) is satisfied. So (ii) is the significant condition in the definition above.
Example 6.5. Let us consider the Denjoy domain \( \Omega := \mathbb{C} \setminus \{ \infty \} \cup \bigcup_{n=1}^{\infty} [2n - 1, 2n] \). It is clear that the ordered sets \( \{2, 3, 6, 7, 10, 11\} \) and \( \{4, 5, \infty, 1\} \) are border sets of \( \partial \Omega \), but \( \{1, 4, 5, \infty\} \) is not.

Definition 6.6. Given a border set of \( \partial \Omega \) with four points, \( A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \), we denote by \( \gamma(A) \) the unique simple closed geodesic in \( \Omega \) which separates \([\alpha_2, \alpha_3] \) from \([\alpha_4, \alpha_1] \) (\( \gamma(A) \) meets \( \mathbb{R} \) just in \((\alpha_1, \alpha_2) \) and \((\alpha_3, \alpha_4) \)).

The characterization of LII in [2] is the following.

Theorem 6.7. ([2, Theorems 4 and 5]) Let \( \Omega \) be a Denjoy domain. Then, \( \Omega \) has LII if and only if \( I \) is strongly uniformly separated in \( \Omega_0 \) and there exists a positive constant \( c \) such that for any border set of \( \partial \Omega_0 \), \( A = \{\alpha_1, \ldots, \alpha_{2n}\} \) with \( n \geq 3 \), we have that

\[
\frac{1}{n} \sum_{j=1}^{n} L_{\Omega_0}(\gamma((\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2}))) > c.
\]

Remark 6.8. At the sight of this characterization of LII, it is clear that we just need to estimate the lengths of simple closed geodesics up to multiplicative constants; however, an additive constant in the estimate would be a “large error”. For this reason, we need that \( A \)-lines and \( B \)-lines would be chord-arc instead of \( (a, b) \)-quasdigeodesics (with \( b > 0 \)).

Furthermore, [2, Theorem 4] provides an estimate of \( L_{\Omega_0}(\gamma((\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2}))) \). Unfortunately, this estimate involves a different Möbius map \( U = U_{\{\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2}\}} \) for each border set, which has a not nice expression (see [2, p. 378]), and there is not an explicit expression for the constants in the estimates. Besides, there are no criteria that guarantees that the set \( I \) is strongly uniformly separated; rather than having a topological condition like “\( B_{\Omega_0}(x, r_0) \) is simply connected”, we would prefer to have a metric condition (especially having at our disposal good results which allow to estimate easily the metric).

Using the results of this paper we obtain an improvement of Theorem 6.7, which solves the inconveniences of the results in [2, Theorem 4]. We have a direct estimate of \( L_{\Omega_0}(\gamma((\alpha_{2j-1}, \alpha_{2j}, \alpha_{2j+1}, \alpha_{2j+2}))) \) (without any Möbius map), by Theorems 5.4 and 5.8.

Let us define first a function \( D_{\Omega} \), if \( \Omega \cap \mathbb{R} = \cup_n (a_n, b_n) \), as follows:

- If \( a, b \in \Omega \cap \mathbb{R} \), we define \( D_{\Omega}(a, b) \) as the function comparable to \( d_{\Omega}(a, b) \) appearing in Theorem 5.3, i.e. \( D_{\Omega}(a, b) := L_{\Omega}([a, a + i|b - a|/2] \cup [b, b + i|b - a|/2]) \).
- If \( a \in \Omega \cap \mathbb{R} \), we define \( D_{\Omega}(a, (a_n, b_n)) \) as the function comparable to \( d_{\Omega}(a, (a_n, b_n)) \) appearing in Theorem 5.6.
- We define \( D_{\Omega}((a_m, b_m), (a_n, b_n)) \) as the function comparable to \( d_{\Omega}(a, (a_n, b_n)) \) appearing in Theorem 5.4.
- If \( a \in (a_m, b_m) \), we also define \( D_{\Omega}(a) := \inf_{n \neq m} D_{\Omega}(a, (a_n, b_n)) \).

Therefore, \( D_{\Omega} \) can be easily estimated by Theorem 5.8.

Now we can state our characterization of LII.

Theorem 6.9. Let \( \Omega \) be a Denjoy domain. Then, \( \Omega \) has LII if and only if there exists a positive constant \( c \) such that:

(i) for any border set of \( \partial \Omega_0 \), \( A = \{\alpha_1, \ldots, \alpha_{2n}\} \) with \( n \geq 3 \), we have that

\[
\frac{1}{n} \sum_{j=1}^{n} D_{\Omega_0}((\alpha_{2j-1}, \alpha_{2j}), (\alpha_{2j+1}, \alpha_{2j+2})) > c,
\]
(ii) $D_{\Omega_0}(x_1, x_2) > c$, for any $x_1, x_2 \in I, $
(iii) $D_{\Omega_0}(x) > c$, for any $x \in I.$

Proof. Theorems 5.3, 5.4 and 5.6 allow to use the simple function $D_\Omega$ instead of $d_\Omega.$

By Theorem 6.7, it is enough to show that the condition “$B_{\Omega}(x, r_0)$ is simply connected for every $x \in I$” is equivalent to (iii). This equivalence is a consequence of the two following facts:

$$\text{sup} \left\{ t > 0 : B_{\Omega_0}(x, r_0) \text{ is simply connected} \right\} = \text{inf} \left\{ L_{\Omega_0}(g) : g \text{ is a loop with base point } x \right\},$$

and a geodesic loop in $\Omega_0$ is not homotopically trivial in $\Omega_0.$

We prove just the second fact since the first one is well known. Let us consider a geodesic loop $\gamma$ with base point $x$, a universal covering map $\pi : \mathbb{D} \rightarrow \Omega_0$, and the lift $\tilde{\gamma}$ of $\gamma$ starting in $\tilde{x} \in \mathbb{D}$. If $\gamma$ is homotopically trivial in $\Omega_0$, then $\tilde{\gamma}$ finishes in $\tilde{x}$ also, i.e. $\tilde{\gamma}$ is a geodesic loop in $\mathbb{D}$, which is a contradiction, since there are no geodesic loop in $\mathbb{D}.$

□

References


Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain

E-mail address: jomaro@math.uc3m.es

Facultad de Matemáticas, Universidad Autónoma de Guerrero, Carlos E. Adame No.54 Col. Garita, 39650 Acapulco Gro., Mexico.

E-mail address: josemariasigarretaalmira@yahoo.es