ZERO LOCATION AND ASYMPTOTIC BEHAVIOR FOR EXTREMAL POLYNOMIALS WITH NON-DIAGONAL SOBOLEV NORMS

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Abstract. In this paper we are going to study the zero location and asymptotic behavior of extremal polynomials with respect to a generalized non-diagonal Sobolev norm in the worst case, i.e., when the quadratic form degenerates. The orthogonal polynomials with respect to this Sobolev norm are a particular case of those extremal polynomials. The multiplication operator by the independent variable is the main tool in order to obtain our results.

Key words and phrases: Multiplication operator; zero location; asymptotic behavior; Sobolev orthogonal polynomials; extremal polynomials; non-diagonal Sobolev norms.

1. Introduction.

Weighted Sobolev spaces are an interesting topic in many fields of Mathematics. In the classical book [19] we can find the point of view of Partial Differential Equations. We are mainly interested in the relationship between this topic and Approximation Theory in general, and Sobolev orthogonal polynomials in particular.

Sobolev orthogonal polynomials have been more and more investigated in recent years. In particular, in [16] and [17], the authors showed that the expansions with Sobolev orthogonal polynomials can avoid the Gibbs phenomenon which appears with classical orthogonal series in $L^2$.

In [32], [33], [34] and [35] the authors developed a theory of general Sobolev spaces with respect to measures in the real line, in order to apply it to the study of Sobolev orthogonal polynomials.

Sobolev orthogonal polynomials on the unit circle and, more generally, on curves is a topic of recent and increasing interest in approximation theory; see, for instance, [5] and [14] (for the unit circle) and [27] and [3] (for the case of Jordan curves). The papers [2], [3], [5], [14], [23] and [25] deal with Sobolev spaces on curves and more general subsets of the complex plane.

One of the central problems in the theory of Sobolev orthogonal polynomials is to determine their asymptotic behavior. In [24] the authors show how to obtain the $n$-th root asymptotic of Sobolev orthogonal polynomials if the zeros of these polynomials are contained in a compact set of the complex plane. Although the uniform bound of the zeros of orthogonal polynomials holds for every measure with compact support in the case without derivatives, it is an open problem to bound the zeros of Sobolev orthogonal polynomials. The boundedness of the zeros is a consequence of the boundedness of the multiplication operator $Mf(z) = zf(z)$; in fact, the zeros of the Sobolev orthogonal polynomials are contained in the disk $\{z : |z| \leq 2\|M\|\}$ (see Theorem 2.1).

In [2], [23], [31], [33], [35], [36] and [37], there are some answers to the question stated in [24] about some conditions for $M$ to be bounded: the more general result on this topic is [2, Theorem 8.1] which characterizes in a simple way (in terms of equivalent norms in Sobolev spaces) the boundedness of $M$ for the classical “diagonal” case

$$\|q\|_{W^k,p(\mu_0,\mu_1,\ldots,\mu_N)} := \left(\sum_{k=0}^{N} \|q^{(k)}\|_{L^p(\mu_k)}^p\right)^{1/p}$$

(see Theorem 2.2 below, which is [2, Theorem 8.1] in the case $N = 1$). The rest of the papers mention several conditions which guarantee the equivalence of norms in Sobolev spaces, and consequently, the boundedness of $M$.

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In the papers [1], [3], [9], [10], [14], [23], [26] and [31] the authors deal with different “non-diagonal” Sobolev norms. In particular, in [3], [14], [23], [26] and [31] the authors study the asymptotic behavior of orthogonal polynomials with respect to non-diagonal Sobolev inner products. In [23] the authors deal with the asymptotic behavior of extremal polynomials with respect to the following non-diagonal Sobolev norms. Given a finite Borel positive measure $\mu$ with compact support $S(\mu)$ consisting of infinitely many points in the complex plane, let us consider the diagonal matrix $A := \text{diag}(\lambda_j), 0 \leq j \leq N$, with $\lambda_j$ positive $\mu$-almost everywhere measurable functions, and $U := (u_{jk}), 0 \leq j, k \leq N$, a matrix of measurable functions such that the matrix $U(x) = (u_{jk}(x)), 0 \leq j, k \leq N$, is unitary $\mu$-almost everywhere. If $V := UAU^*$, where $U^*$ denotes the transpose conjugate of $U$ (note that then $V$ is a positive definite matrix), and $1 \leq p < \infty$, we define the Sobolev norm

$$
\|q\|_{W^{N,p}(\mu)} := \left( \int [q, q', \ldots, q^{(N)}]V^{2/p}(q, q', \ldots, q^{(N)})^{p/2}d\mu \right)^{1/p}.
$$

(1)

It is not difficult to verify that under the assumptions imposed, $\| \cdot \|_{W^{N,p}(\mu)}$ defines a norm on the space of polynomials $\mathbb{P}$. If $U$ is not the identity matrix $\mu$-almost everywhere, then (1) defines a generalized non-diagonal Sobolev norm in which the product of derivatives of different order appears.

We say that $q_n = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ is an $n$-th monic extremal polynomial with respect to the norm (1) if

$$\|q_n\|_{W^{N,p}(\mu)} = \inf \left\{ \|q\|_{W^{N,p}(\mu)} : q = z^n + b_{n-1}z^{n-1} + \cdots + b_1z + b_0, b_j \in \mathbb{C} \right\}.$$

It is clear that there exists at least an $n$-th monic extremal polynomial. Furthermore, it is unique if $1 < p < \infty$ (see e.g. [6, pp. 22-23]). If $p = 2$, then the $n$-th monic extremal polynomial is precisely the $n$-th monic Sobolev orthogonal polynomial with respect to the inner product corresponding to (1).

The following is one of the basic results of [23].

**Theorem 1.1.** ([23, Theorem 1]) *Let $S(\mu)$ be compact and $1 \leq p < \infty$. Assume that there exists a constant $C$ such that

$$\lambda_j \leq C\lambda_k, \quad 0 \leq j, k \leq N,$$

$\mu$-almost everywhere. Let $\{q_n\}_{n \geq 0}$ be a sequence of extremal polynomials with respect to (1). Then the zeros of the polynomials in $\{q_n\}_{n \geq 0}$ are uniformly bounded in the complex plane.*

This result is interesting since it allows the authors to obtain the asymptotic behavior of extremal polynomials (see [23, Theorems 2 and 6]). To require compact support for $\mu$ is a natural hypothesis: if $S(\mu)$ is not bounded, then we can not expect to have zeros uniformly bounded, even in the classical case (orthogonal polynomials in $L^2$).

If we are interested in such a general result as Theorem 1.1, then the main result as Theorem 1.1, but with hypotheses directly on the matrix $V$ (rather than on the diagonal matrix $A$ which appears in its factorization); in exchange for a certain loss of generality, it is required a weaker hypothesis than (2).

Since a wide majority of works about Sobolev spaces (both pure and applied) are focused on the case $N = 1$, we will assume that this is the situation throughout the current paper. That is why we consider directly the weight matrix $V$ as

$$V := \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

where $a, b$ and $c$ are measurable functions, and $V$ is a positive definite matrix $\mu$-almost everywhere. In [31] the authors study the boundedness of the zeros for this norm under the ellipticity hypothesis $|b|^2 \leq (1 - \varepsilon)ac$, $\mu$-almost everywhere for some fixed $0 < \varepsilon \leq 1$.

In this paper, we are interested in obtaining similar results in the worst case, i.e., when the quadratic form $V$ degenerates in an arbitrary set $E$ with $\mu(E) = 0$. In this sense, we replace the useful (but technical) hypothesis $|b|^2 \leq (1 - \varepsilon)ac$ (which avoids the degeneracy of the quadratic form $V$), by a more natural one (in this context) involving integrability properties of the measures. In particular, for the more important case
(\(p = 2\), corresponding to Sobolev orthogonal polynomials) this new condition is \((c \frac{d\mu}{ds})^{-1} \in L^1(\gamma)\), where \(d\mu/ds\) is the Radon-Nykodim derivative of \(\mu\) with respect to the Euclidean length in \(\gamma\), the supporting curve of the measure \(\mu\).

A particular case (simpler to state) of our main result (Theorem 5.4) is the following.

**Theorem 1.2.** Let us fix a rectifiable compact curve \(\gamma\), a finite Borel measure \(\mu\) with compact support \(S(\mu) \subseteq \gamma\), and a positive definite matrix \(\mu\)-almost everywhere, \(V\), defined as in (4). Assume that \((c \frac{d\mu}{ds})^{-1} \in L^1(\gamma)\) and there exists a constant \(C\) such that

\[
c \leq C a,\]

\(\mu\)-almost everywhere. Let \(\{q_n\}_{n \geq 0}\) be the sequence of Sobolev orthogonal polynomials with respect to \(V\mu\). Then the zeros of the polynomials in \(\{q_n\}_{n \geq 0}\) are uniformly bounded in the complex plane.

**Remark 1.3.** Note that the hypothesis \((c \frac{d\mu}{ds})^{-1} \in L^1(\gamma)\) implies that the support of \(\mu\) contains infinitely many points.

In our context there is no such thing as the usual three term recurrence relation for orthogonal polynomials in \(L^2\). Therefore it is very complicated to find an explicit expression for the extremal polynomial of degree \(n\). Hence, it is especially important to count with an asymptotic estimate for the behavior of extremal polynomials.

As an application of Theorem 5.4, we can deduce the asymptotic behavior of extremal polynomials (see Theorems 6.1 and 6.2). In particular, we obtain the \(n\)-th root and the zero counting measure asymptotics both of those polynomials and their derivatives of any order. The asymptotics of the \(n\)-th root is a classical problem in the theory of orthogonal polynomials (see e.g. [23], [24], [25], [39], [40]).

Furthermore, in Theorem 6.2 we find the following asymptotic relation:

\[
\lim_{n \to \infty} \frac{q_n^{(j+1)}(z)}{\omega^{(j)}(z)} = \int \frac{d\omega_{S(\mu)}(x)}{z-x}
\]

for any \(j \geq 0\).

We also have similar results for extremal polynomials in the case \(1 \leq p < 2\); however, since they are not so simple to state as the results for \(p = 2\) (see Theorem 5.3), we prefer not to announce them in this section.

The outline of the paper is as follows. Section 2 is devoted to some results on the multiplication operator and the location of zeros of extremal polynomials. In Section 3 we prove some technical lemmas in order to simplify the proof of Theorem 4.1. We include the proof of Theorem 4.1 in Section 4. We prove a similar result for \(1 \leq p < 2\) in Section 5 and we obtain the results on the multiplication operator. In Section 6 we obtain the results on asymptotic of extremal polynomials.

2. Background and previous results.

In what follows, we will fix a rectifiable compact curve \(\gamma\) in the complex plane, that is not required to be either simple or closed.

As we have mentioned in the introduction, given a finite Borel measure \(\mu\) with compact support \(S(\mu) \subseteq \gamma\) consisting of infinitely many points, and a positive definite matrix \(\mu\)-almost everywhere

\[
V := \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix},
\]

and any polynomial \(q\), we define the Sobolev norm of \(q\) in \(W^{1,2}(a\mu, c\mu)\) by

\[
\|q\|_{W^{1,2}(a\mu, c\mu)} := \left( \int (a|q|^2 + c|q'|^2) d\mu \right)^{1/2}.
\]

It is obviously much easier to deal with this norm than with the one defined in (1). Therefore, one of our main goals is to provide weak hypotheses to guarantee the equivalence of both norms.
In order to bound the zeros of polynomials, one of the most successful strategies has certainly been to bound the multiplication operator by the independent variable $Mf(z) = zf(z)$, where

$$\|M\| := \sup \left\{\|q(z)\|_{W^{1,p}(\mu)} : \|q\|_{W^{1,p}(\mu)} = 1\right\}.$$  

Regarding this issue, the following result is known.

**Theorem 2.1.** ([23, Theorem 3]) Let $S(\mu)$ be compact and $1 \leq p < \infty$. Let $\{q_n\}_{n \geq 0}$ be a sequence of extremal polynomials with respect to (1). Then the zeros of $\{q_n\}_{n \geq 0}$ lie in the bounded disk $\{z : |z| \leq 2\|M\|\}$.

It is also known a simple characterization of the boundedness of $M$.

**Theorem 2.2.** ([2, Theorem 8.1]) Given $1 \leq p < \infty$, the multiplication operator is bounded in $W^{1,p}(a\mu, c\mu)$ if and only if the following condition takes place:

$$
\text{(6)}
\|W^{1,p}(a\mu, c\mu)\| \quad \text{and} \quad \|W^{1,p}(a\mu, c\mu)\|
$$

It is clear that if there exists a constant $C$ such that $c \leq Ca \mu$-almost everywhere, then (6) holds. In [36] and [37] some other very simple conditions implying (6) are shown.

### 3. Technical Lemmas

For the sake of clarity and readability, we have opted for prove all the technical lemmas in this section. This makes the proof of Theorem 4.1 much more understandable.

**Lemma 3.1.** Let $\{s_n\}_n$ and $\{t_n\}_n$ be two sequences of positive numbers. Then

$$
\lim_{n \to \infty} \frac{2s_nt_n}{s_n^2 + t_n^2} = 1 \quad \text{if and only if} \quad \lim_{n \to \infty} \frac{s_n}{t_n} = 1.
$$

**Proof.** Obviously,

$$
\lim_{n \to \infty} \frac{2s_nt_n}{s_n^2 + t_n^2} = \lim_{n \to \infty} \frac{2s_n}{t_n} = \left(\frac{2s_n}{t_n}\right)^2 + 1.
$$

Since the function $f(x) = 2x/(x^2 + 1)$ attains its maximum value, which is 1, only at $x = 1$, the latter limit takes the value 1 if and only if $\lim_{n \to \infty} s_n/t_n = 1$. 

**Lemma 3.2.** Let $\{s_n\}_n, \{s'_n\}_n, \{t_n\}_n, \{t'_n\}_n$ be sequences of positive numbers such that $s_n \leq t_n$ and $s'_n \leq t'_n$ for every $n$. Let us also assume that

$$
\lim_{n \to \infty} \frac{t_n}{t'_n} = \lim_{n \to \infty} \frac{s_n + s'_n}{t_n + t'_n} = 1.
$$

Then,

$$
\lim_{n \to \infty} \frac{s_n}{t_n} = \lim_{n \to \infty} \frac{s'_n}{t'_n} = \lim_{n \to \infty} \frac{s_n}{s'_n} = 1.
$$

**Proof.** By hypothesis we already know that $\lim \inf_{n \to \infty} (s_n/t_n) \leq 1$, but we want to show that it is exactly 1. Seeking for a contradiction, let us assume that there exists a sequence of natural numbers $\{n_k\}_k$ and a real number $c < 1$ such that $\lim_{k \to \infty} (s_{n_k}/t_{n_k}) = c$. Then,

$$
\lim_{k \to \infty} \frac{s_{n_k} + s'_{n_k}}{t_{n_k} + t'_{n_k}} = \lim_{k \to \infty} = \frac{s_{n_k}}{t_{n_k}} + \frac{s'_{n_k}}{t'_{n_k}} \leq \frac{c + \sup_k \frac{s_{n_k}}{t_{n_k}}}{2} \leq \frac{c + 1}{2} < 1,
$$

which contradicts our hypothesis. Therefore, $\lim \inf_{n \to \infty} (s_n/t_n) = 1$; since $s_n \leq t_n$, we deduce $\lim_{n \to \infty} (s_n/t_n) = 1$. Repeating the same argument we can conclude as well that $\lim_{n \to \infty} (s'_n/t'_n) = 1$.

Notice now

$$
\lim_{n \to \infty} \frac{s_n}{s'_n} = \left(\lim_{n \to \infty} \frac{s_n}{t_n}\right) \left(\lim_{n \to \infty} \frac{t_n}{t'_n}\right) \left(\lim_{n \to \infty} \frac{t'_n}{s'_n}\right) = 1.
$$

This finishes the proof. 

In what follows $a, b$ and $c$ refer to the coefficients of the fixed matrix $V$ defined in (4).
We say that \( \{f_n\} \subset \mathcal{P} \) is an extremal sequence if, for every \( n \), \( f_n \) is non-constant, \( \|f_n\|_{L^\infty(\mu)} = 1 \) and
\[
\lim_{n \to \infty} \frac{\left| \int 2bf_nf'_n \, d\mu \right|}{\int (|f_n|^2 + |f'_n|^2) \, d\mu} = 1.
\]

Remark 3.4. Notice that, since \( |b| < \sqrt{ac} \mu \)-almost everywhere,
\[
\left| \int 2bf_nf'_n \, d\mu \right| \leq \int (|a| |f_n|^2 + c |f'_n|^2) \, d\mu
\]
for every \( f \in \mathcal{P} \).

Lemma 3.5. If \( \{f_n\} \) is an extremal sequence, then:
\[
\lim_{n \to \infty} \frac{\left| \int bf_nf'_n \, d\mu \right|}{\sqrt{ac} |f_n| f'_n} = \lim_{n \to \infty} \frac{\int \sqrt{ac} |f_n| f'_n \, d\mu}{\left( \int \sqrt{ac} |f_n| f'_n \, d\mu \right)^{1/2}} = 1.
\]

Proof. First notice that we can rewrite the limit in the definition of extremal sequence as the limit of the following product
\[
\lim_{n \to \infty} \frac{\left| \int bf_nf'_n \, d\mu \right|}{\sqrt{ac} |f_n| f'_n} = \lim_{n \to \infty} \frac{\int \sqrt{ac} |f_n| f'_n \, d\mu}{\left( \int |f_n|^2 \, d\mu \right)^{1/2} \left( \int |f'_n|^2 \, d\mu \right)^{1/2} \left( \int (|a| |f_n|^2 + c |f'_n|^2) \, d\mu \right)} = 1.
\]

Note that each of the three factors above is less than or equal to 1: the first one because \( |b| < \sqrt{ac} \mu \)-almost everywhere; the second one because of the Cauchy-Schwarz inequality and the third one by the fact \( 2xy \leq x^2 + y^2 \). Since the limit of the product is 1, then the limit of every factor must also be 1.

Lemma 3.6. If \( \{f_n\} \) is an extremal sequence, then:
\[
\lim_{n \to \infty} \frac{\int a |f_n|^2 \, d\mu}{\int c |f'_n|^2 \, d\mu} = 1.
\]

Proof. We define
\[
s_n := \left( \int |f_n|^2 \, d\mu \right)^{1/2} \quad \text{and} \quad t_n := \left( \int c |f'_n|^2 \, d\mu \right)^{1/2}.
\]
Since \( V \) is a positive definite matrix \( \mu \)-almost everywhere, the coefficients \( a \) and \( c \) must be positive functions as well. Besides, the support of \( \mu \) consists of infinitely many points, and \( f_n \) is non-constant by definition. Then, \( s_n, t_n > 0 \), and applying now Lemma 3.1 we conclude that
\[
\lim_{n \to \infty} \frac{\int a |f_n|^2 \, d\mu}{\int c |f'_n|^2 \, d\mu} = 1 \quad \text{if and only if} \quad \lim_{n \to \infty} \frac{2 \left( \int a |f_n|^2 \, d\mu \right)^{1/2} \left( \int c |f'_n|^2 \, d\mu \right)^{1/2}}{\int (|a| |f_n|^2 + c |f'_n|^2) \, d\mu} = 1,
\]
and we know that the latter limit is, in effect, 1 by Lemma 3.5.
Definition 3.7. For each $0 < \varepsilon < 1$, we define the sets $A_\varepsilon$ and $A_\varepsilon^c$ as

$$A_\varepsilon := \{ z \in S(\mu) : |b| > (1 - \varepsilon) \sqrt{ac} \},$$

$$A_\varepsilon^c := S(\mu) \setminus A_\varepsilon.$$

Lemma 3.8. If $\{f_n\}_n$ is an extremal sequence and $\varepsilon$ is small enough, then

$$\lim_{n \to \infty} \frac{\int_{A_\varepsilon} |b| |f_n f'_n| \, d\mu}{\int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu} = 1.$$

Remark 3.9. The statement of the lemma might seem strange, because we could have, a priori $\mu(A_\varepsilon) = 0$; however, the existence of the fundamental sequence implies $\mu(A_\varepsilon) > 0$.

Proof. First of all, note that if $\varepsilon \to 0^+$, then $A_\varepsilon^c$ grows up to $S(\mu)$ (except for a zero $\mu$-measured set). Hence, if $\varepsilon$ is small enough, it holds that $\mu(A_\varepsilon^c) > 0$ and the set $A_\varepsilon^c$ has infinitely many points. Consequently, $\int_{A_\varepsilon^c} \sqrt{ac} |f_n f'_n| \, d\mu > 0$ for every $n$.

We will start the proof showing that

$$(7) \quad \lim_{n \to \infty} \frac{\int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu}{\int_{A_\varepsilon^c} \sqrt{ac} |f_n f'_n| \, d\mu} = \infty.$$

By Lemma 3.5 it is known

$$\lim_{n \to \infty} \frac{\left| \int_{A_\varepsilon} b f_n f'_n \, d\mu \right|}{\int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu} = 1$$

and

$$\frac{\left| \int_{A_\varepsilon} b f_n f'_n \, d\mu \right|}{\int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu} = \frac{\int_{A_\varepsilon} |b| |f_n f'_n| \, d\mu + \int_{A_\varepsilon} |b| |f_n f'_n| \, d\mu}{\int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu} \leq \frac{(1 - \varepsilon) \int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu + \int_{A_\varepsilon} |b| |f_n f'_n| \, d\mu}{\int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu + \int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu} \leq (1 - \varepsilon) \int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu + \int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu \leq 1.$$

Hence,

$$\lim_{n \to \infty} \frac{(1 - \varepsilon) \int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu + \int_{A_\varepsilon} |b| |f_n f'_n| \, d\mu}{\int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu + \int_{A_\varepsilon} \sqrt{ac} |f_n f'_n| \, d\mu} = 1.$$
or equivalently
\[
1 - \varepsilon + \frac{\int_{A_{\varepsilon}} |b| |f_n f'_n| \, d\mu}{\int_{A_{\varepsilon}} \sqrt{ac} |f_n f'_n| \, d\mu} \leq \frac{\int_{A_{\varepsilon}} \sqrt{ac} |f_n f'_n| \, d\mu}{\int_{A_{\varepsilon}} \sqrt{ac} |f_n f'_n| \, d\mu} = 1.
\]

Therefore
\[
\lim_{n \to \infty} \frac{\int_{A_{\varepsilon}} |b| |f_n f'_n| \, d\mu}{\int_{A_{\varepsilon}} \sqrt{ac} |f_n f'_n| \, d\mu} = \lim_{n \to \infty} \frac{\int_{A_{\varepsilon}} \sqrt{ac} |f_n f'_n| \, d\mu}{\int_{A_{\varepsilon}} \sqrt{ac} |f_n f'_n| \, d\mu} = \infty,
\]
which gives (7). Furthermore, we can deduce that \( \int_{A_{\varepsilon}} \sqrt{ac} |f_n f'_n| \, d\mu > 0 \) for every \( n \) large enough.

In order to finish the proof, note that
\[
0 \leq \frac{1}{1 - \varepsilon} \int_{A_{\varepsilon}} |b| |f_n f'_n| \, d\mu \leq \frac{\int_{A_{\varepsilon}} |b| |f_n f'_n| \, d\mu}{\int_{A_{\varepsilon}} \sqrt{ac} |f_n f'_n| \, d\mu} \leq \frac{\int_{A_{\varepsilon}} \sqrt{ac} |f_n f'_n| \, d\mu}{\int_{A_{\varepsilon}} \sqrt{ac} |f_n f'_n| \, d\mu}.
\]

This fact and (7) give
\[
\lim_{n \to \infty} \frac{\int_{A_{\varepsilon}} |b| |f_n f'_n| \, d\mu}{\int_{A_{\varepsilon}} |b| |f_n f'_n| \, d\mu} = 0,
\]
which implies the result.

**Lemma 3.10.** If \( \{f_n\}_n \) is an extremal sequence and \( \varepsilon \) is small enough, then
\[
\lim_{n \to \infty} \frac{\int_{A_{\varepsilon}} (a |f_n|^2 + c |f'_n|^2) \, d\mu}{\int_{A_{\varepsilon}} (a |f_n|^2 + c |f'_n|^2) \, d\mu} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{\int_{A_{\varepsilon}} (a |f_n|^2 + c |f'_n|^2) \, d\mu}{\int_{A_{\varepsilon}} (a |f_n|^2 + c |f'_n|^2) \, d\mu} = 0.
\]

**Proof.** Let us notice first that, by Lemma 3.8, \( \int_{A_{\varepsilon}} |b| |f_n f'_n| \, d\mu > 0 \) for every \( n \) large enough. Furthermore, it holds
\[
\frac{\int 2b f_n \overline{f'_n} \, d\mu}{\int (a |f_n|^2 + c |f'_n|^2) \, d\mu} \leq \frac{\int 2 |b| |f_n f'_n| \, d\mu}{\int (a |f_n|^2 + c |f'_n|^2) \, d\mu} \leq \frac{\int 2 |b| |f_n f'_n| \, d\mu}{\int (a |f_n|^2 + c |f'_n|^2) \, d\mu}.
\]

Using now that \( 2 |b| |f_n f'_n| \leq 2 \sqrt{ac} |f_n f'_n| \leq a |f_n|^2 + c |f'_n|^2 \), it is straightforward that
\[
\frac{\int 2b f_n \overline{f'_n} \, d\mu}{\int (a |f_n|^2 + c |f'_n|^2) \, d\mu} \leq \frac{\int 2 |b| |f_n f'_n| \, d\mu}{\int (a |f_n|^2 + c |f'_n|^2) \, d\mu} \leq 1 \quad \text{and} \quad \frac{\int 2 |b| |f_n f'_n| \, d\mu}{\int (a |f_n|^2 + c |f'_n|^2) \, d\mu} \leq 1.
\]
By the definition of extremal sequence we have
\[
\lim_{n \to \infty} \frac{\int |b| |f_n|^{2} \, d\mu}{\int (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu} = 1.
\]

Taking limits in (8) and applying, once again, the definition of extremal sequence and Lemma 3.8 it holds
\[
1 = \lim_{n \to \infty} \frac{\int 2 |b| |f_n|^{2} \, d\mu}{\int (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu} \leq \lim \inf_{n \to \infty} \frac{\int 2 |b| |f_n|^{2} \, d\mu}{\int (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu}
\]
\[
= \lim \inf_{n \to \infty} \frac{\int_{A_{\varepsilon}} 2 |b| |f_n|^{2} \, d\mu}{\int (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu} \leq \lim \sup_{n \to \infty} \frac{\int_{A_{\varepsilon}} 2 |b| |f_n|^{2} \, d\mu}{\int (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu} \leq 1,
\]
and therefore
\[
\lim_{n \to \infty} \frac{\int_{A_{\varepsilon}} 2 |b| |f_n|^{2} \, d\mu}{\int (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu} = 1.
\]

Using all these facts and Lemma 3.8, we can conclude
\[
\lim_{n \to \infty} \frac{\int_{A_{\varepsilon}} (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu}{\int (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu} = \lim_{n \to \infty} \frac{\int 2 |b| |f_n|^{2} \, d\mu}{\int 2 |b| |f_n|^{2} \, d\mu} = 1.
\]

Since \( \int (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu = \int_{A_{\varepsilon}} (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu + \int_{A_{\varepsilon}^c} (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu \), we deduce
\[
\lim_{n \to \infty} \frac{\int_{A_{\varepsilon}} (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu}{\int_{A_{\varepsilon}} (a |f_n|^{2} + c |f'_n|^{2}) \, d\mu} = 0.
\]
\[
\square
\]

**Lemma 3.11.** If \( \{f_n\}_n \) is an extremal sequence and \( \varepsilon \) is small enough, then
\[
\lim_{n \to \infty} \frac{\int_{A_{\varepsilon}} a |f_n|^{2} \, d\mu}{\int a |f_n|^{2} \, d\mu} = 1 \quad \text{and} \quad \lim_{n \to \infty} \frac{\int_{A_{\varepsilon}} c |f'_n|^{2} \, d\mu}{\int c |f'_n|^{2} \, d\mu} = 1.
\]

**Proof.** Let us define the sequences
\[
s_n := \int_{A_{\varepsilon}} a |f_n|^{2} \, d\mu, \quad t_n := \int a |f_n|^{2} \, d\mu, \quad s'_n := \int_{A_{\varepsilon}} c |f'_n|^{2} \, d\mu, \quad t'_n := \int c |f'_n|^{2} \, d\mu.
\]
According to Lemma 3.10 it holds
\[
\lim_{n \to \infty} \frac{s_n + s'_n}{t_n + t'_n} = 1.
\]
By Lemma 3.8, we have for every \( n \) large enough
\[
0 < \int_{A_{\varepsilon}} |b| |f_n|^{2} \, d\mu \leq \int_{A_{\varepsilon}} \sqrt{a^2 c^2 + c^4} |f_n|^{2} \, d\mu \leq \left( \int_{A_{\varepsilon}} a |f_n|^{2} \, d\mu \right)^{1/2} \left( \int_{A_{\varepsilon}} c |f'_n|^{2} \, d\mu \right)^{1/2}
\]
and therefore
\[ \int_{A_ε} a |f_n|^2 dμ, \quad \int_{A_ε} c |f_n'|^2 dμ > 0. \]

Now, according to Lemma 3.2 it will be sufficient to show that
\[ \lim_{n \to \infty} \frac{\int_{A_ε} a |f_n|^2 dμ}{\int_{A_ε} c |f_n'|^2 dμ} = 1, \]
and this holds by Lemma 3.6.

**Lemma 3.12.** If \( \{f_n\}_n \) is an extremal sequence, then for every \( \varepsilon > 0 \) small enough with \( \mu(A_ε^c) > 0 \) and for every \( t \in (0, 1) \) there exists \( N \) such that
\[ \min_{z \in A_ε^c} |f_n(z)| < t \]
for every \( n \geq N \).

**Proof.** Seeking for a contradiction, let us assume that there exists some \( \varepsilon > 0 \) with \( \mu(A_ε^c) > 0 \), some \( t \in (0, 1) \) and a subsequence \( \{n_k\}_k \) with \( |f_{n_k}(z)| \geq t \) for every \( z \in A_ε^c \) and for every \( k \).

Notice first that since \( a > 0 \) \( \mu \)-almost everywhere in \( S(μ) \) and \( \mu(A_ε^c) > 0 \), therefore
\[ \int_{A_ε^c} a dμ > 0. \]

Since \( \|f_{n_k}\|_{L^∞(μ)} = 1 \), it is easy to see that
\[ 0 < \frac{t^2}{\int_{A_ε} a dμ} \leq \frac{\int_{A_ε} a |f_{n_k}|^2 dμ}{\int_{A_ε} a |f_{n_k}|^2 dμ}. \]

However, as consequence of Lemma 3.11 we obtain that
\[ \lim_{n \to \infty} \frac{\int_{A_ε} a |f_{n_k}|^2 dμ}{\int_{A_ε} a |f_{n_k}|^2 dμ} = 0, \]
which gives the contradiction required. \( \square \)

**Definition 3.13.** If \( f \) is a continuous function on \( γ \), we define the oscillation of \( f \) on \( γ \), and we denote it by \( \text{osc}(f) \), as
\[ \text{osc}(f) := \max_{z, w \in γ} |f(z) - f(w)|. \]

**Lemma 3.14.** For \( 1 \leq p < \infty \), let us assume that \( (c dμ/ds)^{-1} \in L^{1/(p-1)}(γ) \), where \( dμ/ds \) is the Radon-Nykodim derivative of \( μ \) with respect to the Euclidean length in \( γ \). (According to our notation, if \( p = 1 \) then \( 1/(p-1) = ∞ \)). Then
\[ \int_γ |f'|^p c dμ \geq k \cdot \text{osc}^p(f), \quad \text{with } 1/k = \left\| \left( \frac{dμ}{ds} \right) \right\|_{L^{1/(p-1)}(γ)}. \]
for every polynomial \( f \).

**Proof.** Let us take \( α, β \in γ \) such that \( \text{osc}(f) = |f(β) - f(α)| \), and let us denote by \([α, β]\) a subcurve of \( γ \) whose endpoints are \( α \) and \( β \). For the sake of simplicity of notation, we will call \( w := c dμ/ds \). Then,
applying the Cauchy-Schwarz inequality when necessary,
\[
|f(\beta) - f(\alpha)| = \left| \int_{[\alpha, \beta]} f'(z) \, dz \right| \leq \int_{[\alpha, \beta]} |f'| \, w^{1/p} w^{-1/p} \, ds
\]
\[
\leq \left( \int_{[\alpha, \beta]} |f'|^p \, w \, ds \right)^{1/p} \left( \int_{[\alpha, \beta]} (w^{-1/p})^{p/(p-1)} \, ds \right)^{(p-1)/p}
\]
\[
\leq \left( \int_{[\alpha, \beta]} |f'|^p \, c \, d\mu \right)^{1/p} \left( \int_{[\alpha, \beta]} w^{-1/(p-1)} \, ds \right)^{(p-1)/p}.
\]
From this inequality we obtain
\[
\int_{[\alpha, \beta]} |f'|^p \, c \, d\mu \geq \int_{[\alpha, \beta]} |f'|^p \, d\mu \geq \frac{|f(\beta) - f(\alpha)|^p}{\left( \int_{[\alpha, \beta]} w^{-1/(p-1)} \, ds \right)^{p-1}}
\]
\[
\geq \left\| \frac{1}{c} \left( c \frac{d\mu}{ds} \right) \right\|_{L^{1/(p-1)}(\gamma)}^{-1} \cdot \text{osc}^p(f).
\]

4. Equivalent norms

We prove now the announced result about the equivalence of the norms. By simplicity of the notation, we start with the case \( p = 2 \).

**Theorem 4.1.** Let us fix a rectifiable compact curve \( \gamma \), a finite Borel measure \( \mu \) with compact support \( S(\mu) \subseteq \gamma \), and a positive definite matrix \( \mu \)-almost everywhere, \( V \), defined as in (4). If \( (c \, d\mu/\, ds)^{-1} \in L^1(\gamma) \), then the norms \( W^{1,2}(V, \mu) \) and \( W^{1,2}(a \mu, c \mu) \) are equivalent on the space of polynomials.

**Proof.** Let us prove that there exists a constant \( C := C(V, \mu) \) such that
\[
C \| f \|_{W^{1,2}(a \mu, c \mu)} \leq \| f \|_{W^{1,2}(V, \mu)} \leq \sqrt{2} \| f \|_{W^{1,2}(a \mu, c \mu)}, \quad \text{for every } f \in \mathbb{P},
\]
where
\[
\| f \|_{W^{1,2}(a \mu, c \mu)}^2 := \int (a |f|^2 + c |f'|^2) \, d\mu \quad \text{and} \quad \| f \|_{W^{1,2}(V, \mu)}^2 := \int (a |f|^2 + c |f'|^2 + 2 \Re(bf \overline{f'})) \, d\mu.
\]

Let us prove first the second inequality \( \| f \|_{W^{1,2}(V, \mu)} \leq \sqrt{2} \| f \|_{W^{1,2}(a \mu, c \mu)} \).

Notice that \( 2 \Re(bf \overline{f'}) \leq |bf'|^2 \leq 2 \sqrt{ac} |f'|^2 \leq a |f|^2 + c |f'|^2 \); therefore, for every polynomial \( f \) it holds
\[
\| f \|_{W^{1,2}(V, \mu)}^2 = \int (a |f|^2 + c |f'|^2 + 2 \Re(bf \overline{f'})) \, d\mu \leq 2 \int (a |f|^2 + c |f'|^2) \, d\mu = 2 \| f \|_{W^{1,2}(a \mu, c \mu)}^2.
\]

In order to prove the first inequality, \( C \| f \|_{W^{1,2}(a \mu, c \mu)} \leq \| f \|_{W^{1,2}(V, \mu)} \), with \( C := C(V, \mu) \), we will seek for a contradiction. Let us assume that there exists a sequence \( \{f_n\}_n \subset \mathbb{P} \) such that
\[
\lim_{n \to \infty} \int (a |f_n|^2 + c |f'_n|^2 + 2 \Re(bf_n \overline{f'_n})) \, d\mu = 0,
\]
which implies
\[
\lim_{n \to \infty} \int 2 \Re(bf_n \overline{f'_n}) \, d\mu = 1.
\]
Given any \( \varepsilon > 0 \), which implies for every \( n \in \mathbb{N} \), we have

\[
\begin{align*}
- \int \mathbb{R}(b f_n') \, d\mu &= \mathbb{R} \left( - \int 2 b f_n' \, d\mu \right) + \left( \int 2 b f_n' \, d\mu \right) \\
&\leq \int (a |f_n|^2 + c |f_n'|^2) \, d\mu \\
&\leq \int (a |f_n|^2 + c |f_n'|^2) \, d\mu \leq 1.
\end{align*}
\]

Hence

\[
\lim_{n \to \infty} \frac{\int 2 b f_n' \, d\mu}{\int (a |f_n|^2 + c |f_n'|^2) \, d\mu} = 1.
\]

If \( f_n \) is constant for some \( n \), then \( \int 2 b f_n' \, d\mu = 0 \); therefore, taking a subsequence if it is necessary, we can assume that \( f_n \) is non-constant for every \( n \).

By homogeneity, without loss of generality we can assume that \( \|f_n\|_{L^\infty(\mu)} = 1 \) for every \( n \). Then \( \{f_n\}_n \) is an extremal sequence.

Applying Lemma 3.6,

\[
(9) \quad \lim_{n \to \infty} \int a |f_n|^2 \, d\mu = 1.
\]

By Lemma 3.12, there exist \( \{z_n\} \subset S(\mu) \) such that \( |f_n(z_n)| \leq 1/2 \) for every \( n \geq N_1 \). Now, taking into account that \( \|f_n\|_{L^\infty(\mu)} = 1 \), we can apply Lemma 3.14, and then

\[
(10) \quad \int |f_n'|^2 \, d\mu \geq k \cdot \text{osc}^2(f_n) \geq k \cdot (\|f_n\|_{L^\infty(\mu)} - |f_n(z_n)|) = k(1 - 1/2)^2 = k/4 > 0
\]

for every \( n \geq N_1 \), with \( 1/k = \|1/|c \, d\mu/ds|\|_{L^1(\gamma)} \).

Let us fix \( \varepsilon \) small enough. On the one hand, by Lemma 3.11 it holds

\[
\int a |f_n|^2 \, d\mu \leq 2 \int_{A_\varepsilon} a |f_n|^2 \, d\mu \leq 2 \|f_n\|_{L^\infty(\mu)} \int_{A_\varepsilon} a \, d\mu = 2 \int_{A_\varepsilon} a \, d\mu
\]

for every \( n \geq N_2 = N_2(\varepsilon) \).

On the other hand, we have

\[
\lim_{\varepsilon \to 0^+} \mu(A_\varepsilon) = \lim_{\varepsilon \to 0^+} \mu(\{|b| \geq (1 - \varepsilon)\sqrt{ac}\}) = \mu(\{|b| \geq \sqrt{ac}\}) = 0,
\]

which implies

\[
\lim_{\varepsilon \to 0^+} \int_{A_\varepsilon} a \, d\mu = 0.
\]

Given any \( \delta > 0 \) there exists \( \varepsilon_0 \) with \( \int_{A_{\varepsilon_0}} a \, d\mu < \delta \). Hence \( \int a |f_n|^2 \, d\mu < 2\delta \) for every \( n \geq N_2(\varepsilon_0) \). Therefore,

\[
\lim_{n \to \infty} \int a |f_n|^2 \, d\mu = 0,
\]

which is a contradiction with (9) and (10).

5. The case \( 1 \leq p < 2 \).

The result for the case \( 1 \leq p < 2 \) can be obtained with a similar argument. We denote the entries of the matrix \( V^{2/p} \) as:

\[
V^{2/p} := \left( \begin{array}{cc} a_p & b_p \\ b_p & c_p \end{array} \right).
\]
$V^{2/p}$ is a positive definite matrix $\mu$-almost everywhere, since $V$ has the same property. We define

$$\|f\|_{W^{1,\infty}(D\mu)} := \left( \int (a_p |f|^2 + c_p |f'|^2)^{p/2} \, d\mu \right)^{1/p}$$

and

$$\|f\|_{W^{1,\infty}(V\mu)} := \left( \int (a_p |f|^2 + c_p |f'|^2 + 2 \Re(b_p f \overline{f'}))^p/2 \, d\mu \right)^{1/p},$$

for every polynomial $f$.

The following result is well known.

**Lemma 5.1.** Let us consider $0 < a \leq 1$. Then

1. $|y^a - x^a| \leq |y - x|^a$ for every $x, y \in \mathbb{R}$,
2. $2^{a-1}(y^a + x^a) \leq (y + x)^a$ for every $x, y \geq 0$.

**Lemma 5.2.** If $1 \leq p \leq 2$ and $\{f_n\}_n$ is an extremal sequence, then:

$$\lim_{n \to \infty} \int \frac{|b_p f_n f'_n|^p/2 \, d\mu}{\left( \int (a_p |f_n|^2)^{p/2} \, d\mu \right)^{1/2}} = \lim_{n \to \infty} \int \left( \frac{\sqrt{a_p c_p} |f_n f'_n|}{\sqrt{a_p |f_n|^2}} \right)^{p/2} \, d\mu = 1,$$

$$\lim_{n \to \infty} \frac{2 \left( \int (a_p |f_n|^2)^{p/2} \, d\mu \right)^{1/2} \left( \int (c_p |f'_n|^2)^{p/2} \, d\mu \right)^{1/2}}{\int \left( \int (a_p |f_n|^2)^{p/2} + (c_p |f'_n|^2)^{p/2} \, d\mu \right)^{1/2}} = \lim_{n \to \infty} \frac{2^{p/2-1} \int (a_p |f_n|^2)^{p/2} + (c_p |f'_n|^2)^{p/2} \, d\mu}{\int (a_p |f_n|^2 + c_p |f'_n|^2)^{p/2} \, d\mu} = 1.$$

**Proof.** First notice that we can rewrite the limit in the definition of extremal sequence as the limit of the following product

$$\lim_{n \to \infty} \int \frac{|b_p f_n f'_n|^p/2 \, d\mu}{\left( \int (a_p |f_n|^2)^{p/2} \, d\mu \right)^{1/2}} \cdot \frac{\left( \int (c_p |f'_n|^2)^{p/2} \, d\mu \right)^{1/2}}{\left( \int (a_p |f_n|^2)^{p/2} + (c_p |f'_n|^2)^{p/2} \, d\mu \right)^{1/2}} = 1.$$

Note that each of the factors above is less than or equal to 1, using that $|b_p|^2 < a_p c_p \, \mu$-almost everywhere, the Cauchy-Schwarz inequality and Lemma 5.1. Since the limit of the product is 1, then the limit of every factor must also be 1.

**Theorem 5.3.** Let us fix a rectifiable compact curve $\gamma$, a finite Borel measure $\mu$ with compact support $S(\mu) \subseteq \gamma$, and a positive definite matrix $\mu$-almost everywhere, $V$, defined as in (4). If $1 \leq p \leq 2$ and $(c_p^p/2 \, d\mu/\, ds)^{1/2} \in L^{1/(p-1)}(\gamma)$, then the norms $W^{1,p}(V\mu)$ and $W^{1,p}(D\mu)$ are equivalent on the space of polynomials.

**Proof.** The proof of this Theorem follows the same scheme as the proof of Theorem 4.1. So, we will seek for a contradiction. Let us assume that there exists a sequence $\{f_n\}_n \subset \mathbb{P}$ such that

$$\lim_{n \to \infty} \int (a_p |f_n|^2 + c_p |f'_n|^2 + 2 \Re(b_p f_n \overline{f'}_n))^p/2 \, d\mu = 0,$$

without loss of generality, we can assume that $f_n$ is non-constant, and $\|f_n\|_{L^{\infty}(\mu)} = 1$ for every $n$. We call fundamental sequence to $\{f_n\}_n$. Since $1 \leq p \leq 2$, Lemma 5.1 gives

$$\int (a_p |f_n|^2 + c_p |f'_n|^2 + 2 \Re(b_p f_n \overline{f'}_n))^p/2 \, d\mu \geq \int (a_p |f_n|^2 + c_p |f'_n|^2)^p/2 \, d\mu - \int |2 b_p f_n f'_n|^p/2 \, d\mu.$$
This right-hand side of the inequality is positive, because $|b_p| < \sqrt{a_p^{-1}}\mu$-almost everywhere. This implies
\[
\lim_{n \to \infty} \frac{\int (a_p |f_n|^2 + c_p |f'_n|^2)^{p/2} \, d\mu - \int |2b_pf_n f'_n|^{p/2} \, d\mu}{\int (a_p |f_n|^2 + c_p |f'_n|^2)^{p/2} \, d\mu} = 0,
\]
and hence
\[
\lim_{n \to \infty} \frac{\int |2b_pf_n f'_n|^{p/2} \, d\mu}{\int (a_p |f_n|^2 + c_p |f'_n|^2)^{p/2} \, d\mu} = 1.
\]
Using Lemma 5.2 from this moment, the proof follows the same scheme as the proof of Theorem 4.1. We just need to rewrite the technical lemmas for $p$ rather than $2$. Recall that, in fact, Lemma 3.14 is already proved for $1 \leq p < \infty$. □

The following result is a direct consequence of Theorems 2.2 and 5.3.

**Theorem 5.4.** Let us fix a rectifiable compact curve $\gamma$, a finite Borel measure $\mu$ with compact support $S(\mu) \subseteq \gamma$, and a positive definite matrix $\mu$-almost everywhere, $V$, defined as in (4). If $1 \leq p \leq 2$ and $(c_p^{p/2} \, d\mu/ds)^{-1} \in L^{1/(p-1)}(\gamma)$, then the multiplication operator is bounded in $W^{1,p}(V\mu)$ if and only if the following condition takes place:
\[
(12) \quad \text{The norms in } W^{1,p}((a_p^{p/2} + c_p^{p/2})\mu) \text{ and } W^{1,p}(a_p^{p/2}\mu, c_p^{p/2}\mu) \text{ are equivalent on } \mathbb{P}.
\]

The latter theorem and Theorem 2.1 give the following result.

**Theorem 5.5.** Let us fix a rectifiable compact curve $\gamma$, a finite Borel measure $\mu$ with compact support $S(\mu) \subseteq \gamma$, and a positive definite matrix $\mu$-almost everywhere, $V$, defined as in (4). Assume that $1 \leq p \leq 2$, $(c_p^{p/2} \, d\mu/ds)^{-1} \in L^{1/(p-1)}(\gamma)$, and (12) takes place. Let $\{q_n\}_{n \geq 0}$ be a sequence of extremal polynomials with respect to (1). Then the zeros of $\{q_n\}_{n \geq 0}$ lie in the bounded disk $\{z : |z| \leq 2\|M\|\}$.

In general, it is not difficult to check whether or not (12) holds. It is clear that if there exists a constant $C$ such that $c_n \leq Ca_n$ $\mu$-almost everywhere, then (12) holds. In [36] and [37] some other very simple conditions implying (12) are shown.

The following direct corollary of Theorem 5.5 is a stronger version of Theorem 1.2.

**Corollary 5.6.** Let us fix a rectifiable compact curve $\gamma$, a finite Borel measure $\mu$ with compact support $S(\mu) \subseteq \gamma$, and a positive definite matrix $\mu$-almost everywhere, $V$, defined as in (4). Assume that $1 \leq p \leq 2$, $(c_p^{p/2} \, d\mu/ds)^{-1} \in L^{1/(p-1)}(\gamma)$, and $c_n \leq Ca_n$, $\mu$-almost everywhere for some constant $C$. Let $\{q_n\}_{n \geq 0}$ be a sequence of extremal polynomials with respect to (1). Then the zeros of $\{q_n\}_{n \geq 0}$ lie in the bounded disk $\{z : |z| \leq 2\|M\|\}$.

6. **Asymptotic of extremal polynomials**

The class $\text{Reg}$ of regular measures is defined in [40]. For measures supported on a compact set of the complex plane, the authors prove that (see Theorem 3.1.1) $\mu \in \text{Reg}$ if and only if
\[
\lim_{n \to \infty} \|Q_n\|_{L^2(\mu)}^{1/n} = \text{cap}(S(\mu)),
\]
where $Q_n$ denotes the $n$-th monic orthogonal polynomial (in the standard sense) with respect to $\mu$, $\| \cdot \|_{L^2(\mu)}$ is the usual norm in the space $L^2(\mu)$ of square integrable functions with respect to $\mu$ and $\text{cap}(S(\mu))$ denotes the logarithmic capacity of $S(\mu)$.

Given a polynomial $q$ whose degree is exactly $n$, we define the normalized zero counting measure of $q$ as
\[
\nu(q) := \frac{1}{n} \sum_{j=1}^{n} \delta_{z_j},
\]
where $z_1, z_2, \cdots, z_n$ are the zeros of $q$ repeated according to their multiplicity, and $\delta_{z_j}$ is the Dirac measure with mass one at the point $z_j$.

By $\omega_{S(\mu)}$ we denote the equilibrium measure of $S(\mu)$.

**Theorem 6.1.** Let us assume that $1 \leq p \leq 2$, $a_p^{1/2} \mu \in \text{Reg}$, $S(\mu)$ is compact and regular with respect to the Dirichlet problem, and $\{q_n\}_{n \geq 0}$ is the sequence of extremal polynomials with respect to $\| \cdot \|_{W^{1,p}(V\mu)}$. Assume also that $(c_p^{1/2} d\mu/ds)^{-1} \in L^{1/(p-1)}(\gamma)$, and (12) takes place. Then,

$$\lim_{n \to \infty} \|q_n^{(j)}\|_{S(\mu)}^{1/n} = \text{cap}(S(\mu)), \quad j \geq 0.$$ 

Furthermore, if $S(\mu)$ has empty interior and its complement is connected, then

$$\lim_{n \to \infty} \nu(q_n^{(j)}) = \omega_{S(\mu)}, \quad j \geq 0.$$ 

in the weak star topology of measures.

**Proof.** It is sufficient to follow the proof offered in [23, Theorem 2], taking into consideration that the hypothesis removed in our theorem are equivalent to the fact that, in our context, the multiplication operator is bounded (see Theorem 2.2) and the norms of $W^{1,p}(V\mu)$ and $W^{1,p}(D\mu)$ are equivalent (see Theorem 5.4).

Let $\Omega$ be the unbounded component of the complement of $S(\mu)$. We denote by $g_{Q}(z; \infty)$ the Green’s function for $\Omega$ with logarithmic singularity at $\infty$. If $S(\mu)$ is regular with respect to the Dirichlet problem, then $g_{Q}(z; \infty)$ is continuous up to the boundary and it can be extended continuously to all $\mathbb{C}$, with value zero on $\mathbb{C} \setminus \Omega$.

**Theorem 6.2.** Let us assume that $1 \leq p \leq 2$, $a_p^{1/2} \mu \in \text{Reg}$, $S(\mu)$ is compact and regular with respect to the Dirichlet problem, and $\{q_n\}_{n \geq 0}$ is the sequence of extremal polynomials with respect to $\| \cdot \|_{W^{1,p}(V\mu)}$. Assume also that $(c_p^{1/2} d\mu/ds)^{-1} \in L^{1/(p-1)}(\gamma)$, and (12) takes place. Then, for each $j \geq 0$,

$$\limsup_{n \to \infty} |q_n^{(j)}(z)|^{1/n} \leq \text{cap}(S(\mu)) e^{g_{Q}(z; \infty)},$$

uniformly on compact subsets of $\mathbb{C}$. Furthermore, for each $j \geq 0$,

$$\lim_{n \to \infty} |q_n^{(j)}(z)|^{1/n} = \text{cap}(S(\mu)) e^{g_{Q}(z; \infty)},$$

uniformly on each compact subset of $\{z : |z| > 2\|M\| \} \cap \Omega$. Finally, if the interior of $S(\mu)$ is empty and its complement connected, we have equality in (13) for all $z \in \mathbb{C}$, except on a set of capacity zero, $S(\omega_{S(\mu)}) \subset \{z : |z| \leq 2\|M\| \}$ and,

$$\lim_{n \to \infty} q_n^{(j+1)}(z) = \int \frac{d\omega_{S(\mu)}(x)}{z-x},$$

uniformly on each compact subset of $\{z : |z| > 2\|M\| \}$.

**Proof.** The proof in [23, Theorem 6] is valid taking into account again that, in our context, the multiplication operator is bounded (see Theorem 2.2) and the norms of $W^{1,p}(V\mu)$ and $W^{1,p}(D\mu)$ are equivalent (see Theorem 5.4).

**References**


