On the hyperbolicity constant in graphs

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Abstract

If \(X\) is a geodesic metric space and \(x_1, x_2, x_3 \in X\), a geodesic triangle \(T = \{x_1, x_2, x_3\}\) is the union of the three geodesics \([x_1 x_2], [x_2 x_3]\) and \([x_3 x_1]\) in \(X\). The space \(X\) is \(\delta\)-hyperbolic (in the Gromov sense) if any side of \(T\) is contained in a \(\delta\)-neighborhood of the union of the two other sides, for every geodesic triangle \(T\) in \(X\).

We denote by \(\delta(X)\) the sharp hyperbolicity constant of \(X\), i.e. \(\delta(X) := \inf\{\delta \geq 0 : X\ is\ \delta\text{-hyperbolic}\}\). In this paper we obtain several tight bounds for the hyperbolicity constant of a graph and precise values of this constant for some important families of graphs. In particular, we investigate the relationship between the hyperbolicity constant of a graph and its edge number, diameter and cycles. As a consequence of the study, we show that if \(G\) is any graph with \(m\) edges with lengths \(\{l_k\}_{k=1}^m\), then \(\delta(G) \leq \sum_{k=1}^m l_k/4\), and \(\delta(G) = \sum_{k=1}^m l_k/4\) if and only if \(G\) is isomorphic to \(C_m\).

Moreover, we prove the inequality \(\delta(G) \leq \frac{1}{2} \text{diam} G\) for every graph, and we use this inequality in order to compute the precise value \(\delta(G)\) for many graphs.

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1
1 Introduction

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance \[2, 10, 11, 12, 19, 20, 21, 22, 23, 26, 28, 30, 32\].

The theory of Gromov’s spaces was used initially for the study of finitely generated groups, where it was demonstrated to have an enormous practical importance. This theory was applied principally to the study of automatic groups (see \[27\]), that play an important role in sciences of the computation. Another important application of these spaces is secure transmission of information by Internet (see \[19, 20, 21, 22, 23\]). In particular, the hyperbolicity also plays an important role in the spread of viruses through the network (see \[20, 21\]). The hyperbolicity is also useful in the study of DNA data (see \[10\]).

In recent years several investigators have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Klein-Hilbert and Kobayashi metrics are Gromov hyperbolic (under particular conditions on the domain of definition, see \[5, 24, 3\]); the Gehring-Osgood \(j\)-metric is Gromov hyperbolic; and the Vuorinen \(j\)-metric is not Gromov hyperbolic except in the punctured space (see \[14\]). Also, in \[25\] the hyperbolicity of the conformal modulus metric \(\mu\) and the related so-called Ferrand metric \(\lambda^*\), are studied. The study of the Gromov hyperbolicity of the quasihyperbolic and the Poincaré metrics is the subject of \[1, 4, 7, 15, 16, 17, 18, 28, 29, 30, 31, 32\]. In particular, in \[28, 30, 32\] it is proved the equivalence of the hyperbolicity of Riemann surfaces (with their Poincaré metrics) and the hyperbolicity of a simple graph; hence, it is useful to know hyperbolicity criteria for graphs.

In our study on the hyperbolicity constant in graphs we use the notations of \[13\]. We give now the basic facts about Gromov’s spaces. If \(\gamma : [a, b] \rightarrow X\) is a continuous curve in a metric space \((X, d)\), we can define the length of \(\gamma\) as

\[
L(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \cdots < t_n = b \right\}.
\]

We say that \(\gamma\) is a geodesic if it is an isometry, i.e., \(L(\gamma|_{[s,t]}) = d(\gamma(s), \gamma(t)) = |t - s|\) for every \(s, t \in [a, b]\). We say that \(X\) is a geodesic metric space if for every \(x, y \in X\) there exists a geodesic joining \(x\) and \(y\); we denote by \([xy]\) any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If \(X\) is a graph, we use the classical notation \([u, v]\) for the edge of a graph joining the vertices \(u\) and \(v\).

Throughout the paper we just consider graphs which are connected and locally finite (i.e., in each ball there are just a finite number of edges). These conditions guarantee that the graph is a geodesic space (since we consider that every point in any edge of a graph \(G\) is a point of \(G\), whether or not it is a vertex of \(G\)). We allow loops and multiple edges in the graphs. We also allow edges of arbitrary lengths.
If \( X \) is a geodesic metric space and \( J = \{J_1, J_2, \ldots, J_n\} \), with \( J_j \subseteq X \), we say that \( J \) is \( \delta \)-thin if for every \( x \in J_i \) we have that \( d(x, \bigcup_{j \neq i} J_j) \leq \delta \). We denote by \( \delta(J) \) the sharp thin constant of \( J \), i.e. \( \delta(J) := \inf\{\delta \geq 0 : J \text{ is } \delta\text{-thin}\} \). If \( x_1, x_2, x_3 \in X \), a geodesic triangle \( T = \{x_1, x_2, x_3\} \) is the union of the three geodesics \([x_1x_2], [x_2x_3] \text{ and } [x_3x_1] \). The space \( X \) is \( \delta \)-hyperbolic (or satisfies the \textit{Rips condition} with constant \( \delta \)) if every geodesic triangle in \( X \) is \( \delta \)-thin. We denote by \( \delta(X) \) the sharp hyperbolicity constant of \( X \), i.e. \( \delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X \} \). We say that \( X \) is \textit{hyperbolic} if \( X \) is \( \delta \)-hyperbolic for some \( \delta \geq 0 \). If \( X \) is hyperbolic, then \( \delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\} \).

The hyperbolicity constant \( \delta(X) \) of a geodesic metric space can be viewed as a measure of how “tree-like” the space is, since those spaces with \( \delta(X) = 0 \) are precisely the metric trees.

Remark 1. Any bigon, i.e., a triangle with two equal vertices, in a \( \delta \)-hyperbolic space is obviously \( \delta \)-thin. Note that any geodesic polygon with \( n \geq 3 \) sides in a \( \delta \)-hyperbolic space is \((n - 2)\delta\)-thin (it is enough to decompose the polygon as a union of triangles).

Remark 2. There are several definitions of Gromov hyperbolicity (see e.g. [13]). These different definitions are equivalent in the sense that if \( X \) is \( \delta_A \)-hyperbolic with respect to the definition \( A \), then it is \( \delta_B \)-hyperbolic with respect to the definition \( B \), and there exist universal constants \( c_1, c_2 \) such that \( c_1 \delta_A \leq \delta_B \leq c_2 \delta_A \) (see e.g. [13, p. 41]). However, for a fixed \( \delta \geq 0 \), the set of \( \delta \)-hyperbolic graphs with respect to the definition \( A \), is different, in general, from the set of \( \delta \)-hyperbolic graphs with respect to the definition \( B \). We have chosen this definition since it has a deep geometric meaning (see e.g. [13, Chapter 3]).

Remark 3. Some authors (see e.g. [10]) consider just geodesic triangles in any graph \( G \) which have vertices in \( V(G) \); by doing so we obtain a definition which is equivalent (in the sense of Remark 2) to our definition if every edge in \( G \) has length 1. However, if we want to deal with graphs with edges of arbitrary length, we must consider geodesic triangles with vertices in \( G \).

We would like to point out that deciding whether or not a space is hyperbolic is usually extraordinarily difficult: Notice that, first of all, we have to consider an arbitrary geodesic triangle \( T \), and calculate the minimum distance from an arbitrary point \( P \) of \( T \) to the union of the other two sides of the triangle to which \( P \) does not belong to. And then we have to take supremum over all the possible choices for \( P \) and then over all the possible choices for \( T \). It means that if our space is, for instance, an \( n \)-dimensional manifold and we select two points \( P \) and \( Q \) on different sides of a triangle \( T \), the function \( F \) that measures the distance between \( P \) and \( Q \) is a \((3n + 2)\)-variable function (\( 3n \) variables describe the three vertices of \( T \) and two variables describe the points \( p \) and \( q \) in the closed curve given by \( T \)). In order to prove that our space is hyperbolic we would have to take the minimum of \( F \) over the variable that describes \( Q \), and then the supremum over the remaining \( 3n + 1 \) variables, or at least to prove that it is finite. Without disregarding the difficulty of solving a \((3n + 2)\)-variable minimax problem, notice that the main obstacle is that we do not even know in an approximate way the location of geodesics in the space.
Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces. A map \(f : X \rightarrow Y\) is said to be an \((\alpha, \beta)\)-quasi-isometry, with \(\alpha \geq 1, \beta \geq 0\), if for every \(x, y \in X\):
\[
\alpha^{-1}d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.
\]

We say that \(f\) is a quasi-isometry if we do not take into account the constants \(\alpha\) and \(\beta\).

When \(\alpha = 1\) and \(\beta = 0\), \(f\) is said to be an isometry.

A quasi-isometry, in general, is not continuous as we can see in the following example.

The function \(f : \mathbb{R} \rightarrow \mathbb{R}\) defined by \(f(x) = [x]\) is a \((1, 1)\)-quasi-isometry but \(f\) is not continuous in \(\mathbb{N}\).

Let \(X\) be a metric space, \(Y\) and \(Z\) two non empty parts of \(X\) and \(\varepsilon\) a real positive number.

We call \(\varepsilon\)-neighborhood of \(Y\) in \(X\), denoted by \(V_{\varepsilon}(Y)\) to the set \(\{x \in X : d_X(x, Y) \leq \varepsilon\}\).

Two metric spaces \(X\) and \(Y\) are quasi-isometric if there exist a quasi-isometry \(f : X \rightarrow Y\) and a real number \(\varepsilon \geq 0\) such that \(f(X)\) is \(\varepsilon\)-full in \(Y\), i.e., \(V_{\varepsilon}(f(X)) = Y\). An \((\alpha, \beta)\)-quasigeodesic of a metric space \(X\) is a \((\alpha, \beta)\)-quasi-isometry \(\gamma : I \rightarrow X\), where \(I\) is an interval of \(\mathbb{R}\).

If \(D\) is a closed subset of \(X\), we always consider in \(D\) the inner metric obtained by the restriction of the metric in \(X\), that is

\[
d_D(z, w) := \inf \{L_X(\gamma) : \gamma \subset D \text{ is a continuous curve joining } z \text{ and } w\} \geq d_X(z, w).
\]

Consequently, \(L_D(\gamma) = L_X(\gamma)\) for every curve \(\gamma \subset D\).

The following are interesting examples of hyperbolic spaces. \(\mathbb{R}\) is 0-hyperbolic: in fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore we can conclude that \(\mathbb{R}\) is 0-hyperbolic. \(\mathbb{R}^2\) is not hyperbolic: it is clear that the triangles can be drawn with arbitrarily large diameter, so that with the Euclidean metric we may obtain that \(\mathbb{R}^2\) is not hyperbolic. This argument can be generalized in a similar way to higher dimensions: a normed vector space \(E\) is hyperbolic if and only if \(\dim E = 1\). Every arbitrary length metric tree is 0-hyperbolic. In fact, any point of a geodesic triangle in a tree belongs simultaneously to two sides of the triangle. Every bounded metric space \(X\) is \((\text{diam } X)\)-hyperbolic. Every simply connected complete Riemannian manifold with sectional curvature verifying \(K \leq -k^2\), for some positive constant \(k\), is hyperbolic. We refer to \([6, 8, 9, 13]\) for more background and further results.

## 2 Hyperbolicity constant in graphs

Since it is not easy to guarantee the hyperbolicity, it is interesting to relate the hyperbolicity constant with other important parameters of a graph or with some properties of the graph.

We start with some results which relate hyperbolicity with local hyperbolicity. We say that a sequence of closed sets \(\{K_n\}_n\) in a metric space \(X\) is an exhaustion of \(X\) if \(K_n \subseteq K_{n+1}\) for every \(n\) and given any compact set \(K \subset X\), there exists \(N\) with \(K \subseteq K_N\).
Theorem 4. Assume that there exist $\delta \geq 0$ and an exhaustion $\{K_n\}_n$ of a geodesic metric space $X$ such that $K_n$ is $\delta$-hyperbolic for every $n$. Then $X$ is $\delta$-hyperbolic.

Proof. Let $T = [xy] \cup [yz] \cup [zx]$ be any geodesic triangle in $X$, and let $u \in [xy]$. $\mathcal{V}_\delta(T)$ is contained in $K_N$ for some $N$. Since $T$ is a geodesic triangle in $X$, it is also a geodesic triangle in $K_N$. Since $K_N$ is $\delta$-hyperbolic, there exists $v \in [yz] \cup [zx]$ such that $d_X(u,v) \leq d_{K_N}(u,v) \leq \delta$.

With the same aim, we relate the hyperbolicity of a graph with the hyperbolicity of its subgraphs.

We say that a subgraph $\Gamma$ of $G$ is isometric if $d_{\Gamma}(x,y) = d_G(x,y)$ for every $x,y \in \Gamma$.

Lemma 5. If $\Gamma$ is an isometric subgraph of $G$, then $\delta(\Gamma) \leq \delta(G)$.

Proof. Note that, by hypothesis, $d_{\Gamma}(x,y) = d_G(x,y)$ for every $x,y \in \Gamma$; therefore, every geodesic triangle in $\Gamma$ is a geodesic triangle in $G$. Hence, $\delta(\Gamma) \leq \delta(G)$.

In [13] we can find the following result.

Lemma 6 (Invariance of hyperbolicity). Let $f : X \longrightarrow Y$ be an $(\alpha, \beta)$-quasi-isometry between two geodesic metric spaces. If $Y$ is $\delta$-hyperbolic, then $X$ is $\delta'$-hyperbolic, where $\delta'$ is a constant which just depends on $\delta$, $\alpha$ and $\beta$.

Besides, if $f$ is $\varepsilon$-full for some $\varepsilon \geq 0$, then $X$ is hyperbolic if and only if $Y$ is hyperbolic. Furthermore, if $X$ is $\delta'$-hyperbolic, then $Y$ is $\delta$-hyperbolic, where $\delta$ is a constant which just depends on $\delta'$, $\alpha$, $\beta$ and $\varepsilon$.

Theorem 7. Assume that $\Gamma$ is a subgraph of a graph $G$ such that there exist $\alpha \geq 1$ and $\beta \geq 0$ with $d_{\Gamma}(x,y) \leq \alpha d_G(x,y) + \beta$, for every $x,y \in \Gamma$. If $G$ is hyperbolic, then $\Gamma$ is hyperbolic. Moreover, if there exists a constant $c$ such that every connected component $E$ of $G \setminus \Gamma$ satisfies $diam_G E \leq c$, then $G$ is hyperbolic if and only if $\Gamma$ is hyperbolic.

Proof. By Lemma 6, it suffices to note that the inclusion $i : \Gamma \longrightarrow G$ is an $(\alpha, \beta)$-quasi-isometry, since $d_G(x,y) \leq d_{\Gamma}(x,y)$ for every $x,y \in \Gamma$. Furthermore, if every connected component $E$ of $G \setminus \Gamma$ satisfies $diam_G E \leq c$, then $i$ is $c$-full.

The next result relates $\delta$ with an important parameter of a graph: the diameter. It is a simple but useful result.

Theorem 8. In any graph $G$ the inequality $\delta(G) \leq \frac{1}{2} diam G$ holds, and furthermore, it is sharp.

Proof. Let us consider a geodesic side $\gamma$ in any geodesic triangle $T \subset G$. Denote by $x, y$ the endpoints of $\gamma$, and by $\gamma_1, \gamma_2$ the other sides of $T$. For any $p \in \gamma$, it is clear that

$$d(p, \gamma_1 \cup \gamma_2) \leq d(p, \{x, y\}) \leq \frac{1}{2} L(\gamma) \leq \frac{1}{2} diam G,$$

and consequently, $\delta(G) \leq \frac{1}{2} diam G$. 


The equality in Theorem 8 is attained by many graphs, as shows the following Theorem. We will need also the following result (see [30]):

**Lemma 9.** Let us consider a geodesic metric space X. If every geodesic triangle in X which is a simple closed curve, is δ-thin, then X is δ-hyperbolic.

This lemma has the following direct consequence.

**Corollary 10.** In any geodesic metric space X,

\[ \delta(X) = \sup \{ \delta(T) : T \text{ is a geodesic triangle which is a simple closed curve} \} . \]

**Theorem 11.** The following graphs with edges of length 1 have these precise values of δ:

- The path graphs verify \( \delta(P_n) = 0 \) for every \( n \geq 1 \).
- The cycle graphs verify \( \delta(C_n) = n/4 \) for every \( n \geq 3 \).
- The complete graphs verify \( \delta(K_1) = \delta(K_2) = 0 \), \( \delta(K_3) = 3/4 \), \( \delta(K_n) = 1 \) for every \( n \geq 4 \).
- The complete bipartite graphs verify \( \delta(K_{1,1}) = \delta(K_{1,2}) = \delta(K_{2,1}) = 0 \), \( \delta(K_{m,n}) = 1 \) for every \( m, n \geq 2 \).
- The Petersen graph \( P \) verifies \( \delta(P) = 3/2 \).
- The wheel graph with \( n \) vertices \( W_n \) verifies \( \delta(W_4) = \delta(W_5) = 1 \), \( \delta(W_n) = 3/2 \) for every \( 7 \leq n \leq 10 \), and \( \delta(W_n) = 5/4 \) for \( n = 6 \) and for every \( n \geq 11 \).

Furthermore, the graphs \( C_n \) and \( K_n \) for every \( n \geq 3 \), \( K_{m,n} \) for every \( m, n \geq 2 \), the Petersen graph \( P \) and \( W_n \) for every \( 4 \leq n \leq 10 \), verify \( \delta(G) = \frac{1}{2} \text{diam } G \).

**Proof.** It is clear that \( \delta(P_n) = 0 \), \( \delta(K_1) = \delta(K_2) = 0 \) and \( \delta(K_{1,1}) = \delta(K_{1,2}) = \delta(K_{2,1}) = 0 \), since these graphs are trees.

Since \( \text{diam } C_n = n/2 \), Theorem 8 gives that \( \delta(C_n) \leq n/4 \). Let us consider a bigon with two vertices \( \{x, y\} \) at distance \( n/2 \), with sides \( \gamma_1, \gamma_2 \) such that \( \gamma_1 \cup \gamma_2 = C_n \). The midpoint \( p \) of \( \gamma_1 \) satisfies \( d(p, \gamma_2) = d(p, \{x, y\}) = n/4 \). Consequently, \( \delta(C_n) = n/4 \). We also have \( \delta(K_3) = 3/4 \), since \( K_3 = C_3 \).

If \( n \geq 4 \), then the diameter of the complete graphs \( K_n \) is \( \text{diam } K_n = 2 \). Therefore, Theorem 8 gives that \( \delta(K_n) \leq 1 \). Consider a simple cycle \( g \) of length 4 in \( K_n \). Fix a point \( x \) in the midpoint of a fixed edge of \( g \); let us consider the point \( y \in g \) at distance 2 from \( x \) and the bigon with vertices \( \{x, y\} \) and sides \( \gamma_1, \gamma_2 \) such that \( \gamma_1 \cup \gamma_2 = g \). The midpoint \( p \) of \( \gamma_1 \) satisfies \( d(p, \gamma_2) = d(p, \{x, y\}) = 1 \). Hence, \( \delta(K_n) = 1 \).

The argument for \( K_{m,n} \), with \( m, n \geq 2 \), is similar to this last one.
Let us fix two non-adjacent points $x, y$ in the “exterior” pentagon $P_0$ of the Petersen graph $P$ and consider the path with length three $g_1 \subset P_0$ joining $x$ and $y$. Let $g_2$ be the path with length three not contained in $P_0$ joining $x$ and $y$. Let $p$ be the midpoint of $g_1$. Then we have $\delta(P) \geq d(p, g_2) = d(p, \{x, y\}) = 3/2$.

Note that $\text{diam} V(P) = 2$. Given two points $p_1, p_2 \in P$, let us denote by $v_i$ a vertex with $d(p_i, v_i) \leq 1/2$ for $i = 1, 2$. Then $d(p_1, p_2) \leq d(p_1, v_1) + \text{diam} V(P) + d(p_2, v_2) \leq 1/2 + 2 + 1/2 = 3$, and $\text{diam} P \leq 3$. Hence, Theorem 8 gives that $\delta(W_n) \geq 2n - 1$, and we deduce $\delta(P) = 3/2$ and $\text{diam} P = 3$.

The wheel graph $W_4$ is isometric to $K_4$, and then $\delta(W_4) = 1$. Theorem 8 gives that $\delta(W_n) \leq 1/2 \text{diam} W_n$. It is not difficult to check that $\text{diam} W_4 = \text{diam} W_5 = 2$, $\text{diam} W_6 = 5/2$ and $\text{diam} W_7 = 3$ for every $n \geq 7$. Since $W_5$ contains a simple cycle with length 4, then $\delta(W_5) \geq 1$; since $\delta(W_5) \leq 1/2 \text{diam} W_5 = 1$, we conclude that $\delta(W_5) = 1$.

Let us consider the simple cycle $C$ in $W_n$ with length $n - 1$ containing every vertex minus the central vertex.

Let $x$ be the midpoint of any edge in $C$, and consider the points $y$ and $z$ in $C$ at distances $(n - 1)/2$ and $(n - 1)/4$, respectively, from $x$. Then $T := \{x, y, z\}$ is a geodesic triangle with $[xy] \cup [yz] \cup [zx] = C$ if $n \in \{6, 7\}$ (recall that $\text{diam} W_6 = 5/2$ and $\text{diam} W_7 = 3$).

The midpoint $p$ of $[xy]$ verifies $d(p, [yz] \cup [zx]) = d(p, \{x, y\}) = (n - 1)/4$, and consequently, $\delta(W_n) \geq (n - 1)/4$ if $n \in \{6, 7\}$. Therefore, $\delta(W_6) \geq 5/4$ and $\delta(W_7) \geq 3/2$. Since $\text{diam} W_6 = 5/2$ and $\text{diam} W_7 = 3$, we have that $\delta(W_6) = 5/4$ and $\delta(W_7) = 3/2$.

Let $x$ be the midpoint of any edge in $C$, and consider the points $y$ and $z$ in $C$ at distances 3 and $(n - 4)/2$, respectively, from $x$ in $C$. Then $T := \{x, y, z\}$ is a geodesic triangle with $[xy] \cup [yz] \cup [zx] = C$ if $n \in \{8, 9, 10\}$ (recall that $\text{diam} W_n = 3$ for every $n \geq 7$). The midpoint $p$ of $[xy]$ verifies $d(p, [yz] \cup [zx]) = d(p, \{x, y\}) = 3/2$, and consequently, $\delta(W_n) \geq 3/2$ if $n \in \{8, 9, 10\}$. Since $\text{diam} W_n = 3$ for every $n \geq 7$, we have that $\delta(W_n) = 3/2$ if $n \in \{8, 9, 10\}$.

If $n \geq 11$, then the simple cycle $C$ in $W_n$ has length $n - 1 \geq 10$, and it is not a geodesic triangle, since any geodesic $\gamma$ verifies $L(\gamma) \leq \text{diam} W_n = 3$. Let us consider the simple cycle $C'$ in $W_n$ with length 9 containing eight consecutive vertices in $C$ and the central vertex $v_0$ in $W_n$. Let $x$ be the point in $C'$ at distance 9/2 from $v_0$. Consider the points $y$ and $z$ in $C'$ at distance 3 from $v_0$. Then $T := \{x, y, z\}$ is a geodesic triangle with $[xy] \cup [yz] \cup [zx] = C'$, since $n \geq 11$. The point $q$ in $[xy]$ with $d(p, x) = 5/4$ verifies $d(p, [yz] \cup [zx]) = d(p, x) = 5/4$, and consequently, $\delta(W_n) \geq \delta(T) \geq 5/4$ if $n \geq 11$. We are proving that this triangle is, in fact, an extremal triangle.

Let us consider any geodesic triangle $T = \{x, y, z\}$ in $W_n$ with $n \geq 11$. By Corollary 10, we can assume that $T$ is also a simple cycle. Since the simple cycle $T$ is not $C$, then it must be a simple cycle $C''$ in $W_n$ with length $m \geq 3$ containing $m - 1$ consecutive vertices in $C$ (which we will call $v_1, \ldots, v_{m-1}$) and the central vertex $v_0$ in $W_n$. Note that $m \leq 9$, since any geodesic $\gamma$ verifies $L(\gamma) \leq \text{diam} W_n = 3$.

Assume first that $x = v_0$ is a vertex of $T$. Since every point $a \in W_n$ verifies $d(a, v_0) \leq 3/2$, then $L([xy]), L([xz]) \leq 3/2$ and hence, $d(p_1, [xz] \cup [yz]) \leq d(p_1, \{x, y\}) \leq 3/4$ for every
that it is not easy to characterize the graphs verifying 
\[ 4 \leq C \] for every graph \( G \). Without loss of 
generality we can assume that \( d(y, v_1) \leq d(z, v_1) \).

If \( d(y, v_0) < 1 \), let us denote by \( y' \) the point with \( y' \in [v_2, v_3] \) and \( d(y, y') = 2 \). Then
\[ z \in [v_1, v_2] \cup [v_2y'], \] since
\begin{align*}
\delta(y, y') &= d(y, v_1) + d(v_1, v_2) + d(v_2, y') = 2, \\
d(v_2, y') &= 1 - d(y, v_1), \\
d(v_3, y') &= 1 - d(v_2, y') = d(y, v_1) = 1 - d(y, v_0), \\
d(y, y') &= d(y, v_0) + d(v_0, v_3) + d(v_3, y') = 2;
\end{align*}
therefore, \( L([yz]) \leq 2 \) and \( d(p_3, [xy] \cup [xz]) \leq d(p_3, \{y, z\}) \leq 1 \) for every \( p_3 \in [yz] \).

If \( 1 \leq d(y, v_0) < 3/2 \), then \( y \in [v_1, v_2] \); let us denote by \( y'' \) the point with \( y'' \in [v_3, v_4] \) and \( d(y, y'') = 5/2 \). Therefore, \( z \in [yv_2] \cup [v_3, v_4] \cup [v_3y''] \), since
\begin{align*}
\delta(y, y'') &= d(y, v_2) + d(v_2, v_3) + d(v_3, y'') = 5/2, \\
d(v_3, y'') &= 3/2 - d(y, v_2), \\
d(v_4, y'') &= 1 - d(v_3, y'') = 1 - 3/2 + d(y, v_2) = 1/2 - d(y, v_1), \\
d(y, y'') &= d(y, v_1) + d(v_1, v_0) + d(v_0, v_4) + d(v_4, y'') = 5/2.
\end{align*}
Hence, \( L([yz]) \leq 5/2 \) and \( d(p_3, [xy] \cup [xz]) \leq 5/4 \) for every \( p_3 \in [yz] \).

If \( d(y, v_0) = 3/2 \), then \( y \) is the midpoint of \([v_1, v_2]\); let us denote by \( y''' \) the midpoint of \([v_4, v_5]\). Since \( d(y, y''') = 3 = \text{diam } W_n \), we have that \( z \in [yv_2] \cup [v_3, v_4] \cup [v_4y'''] \). If \( p_3 \in [yz] \) verifies \( d(p_3, v_3) \geq 1/4 \), then \( d(p_3, [xy] \cup [xz]) \leq d(p_3, \{y, z\}) \leq 3/2 - 1/4 = 5/4 \). If \( p_3 \in [yz] \) verifies \( d(p_3, v_3) \leq 1/4 \), then \( d(p_3, [xy] \cup [xz]) \leq d(p_3, v_0) \leq d(p_3, v_3) + d(v_3, v_0) \leq 1/4 + 1 = 5/4 \).

Hence, if \( v_0 \) is a vertex of \( T \), we have proved that \( \delta(T) \leq 5/4 \). If \( v_0 \) is not a vertex of \( T \), a similar argument gives also \( \delta(T) \leq 5/4 \). Therefore, \( \delta(W_n) \leq 5/4 \) for every \( n \geq 11 \). Hence, \( \delta(W_n) = 5/4 \) for every \( n \geq 11 \).

Finally, since we know the hyperbolicity constants of these graphs, it is direct to check that the graphs \( C_n \) and \( K_n \) for every \( n \geq 3 \), \( K_{m,n} \) for every \( m, n \geq 2 \), and \( \delta(W_n) \) for every \( 4 \leq n \leq 10 \), verify \( \delta(G) = \frac{1}{2} \text{diam } G \).

It is interesting to remark the unexpected behavior of \( \delta(W_n) \). This illustrates the difficulty of the study of the hyperbolicity constant. The final conclusion of Theorem 11 shows that it is not easy to characterize the graphs verifying \( \delta(G) = \frac{1}{2} \text{diam } G \) (even if \( G \) has every edge with length 1).

We are interested in other classes of graphs for which we have \( \delta(G) = \frac{1}{2} \text{diam } G \).

**Theorem 12.** Let \( C_{a,b,c} \) be the graph with two vertices and three edges joining them with lengths \( a \leq b \leq c \). Then \( \delta(C_{a,b,c}) = \frac{c + \min\{b, 3a\}}{4} \).
Proof. Let us denote by \( x_1, x_2 \) the vertices of \( C_{a,b,c} \), and by \( A, B, C \) the edges with lengths \( a, b, c \), respectively.

Assume first that \( b \leq 3a \). Let \( x \) be the point in \( C \) with \( d(x, x_1) = (c + a)/2 \) and \( y \) be the point in \( B \) with \( d(y, x_1) = (b - a)/2 \). Consider the geodesics \([x x_1] \subset C \), \([x_1 y] \subset B \) and 
\([y y_1] = [x x_1] \cup [x_1 y] \). Note that \( L([xy]) = (c + b)/2 \). Let \( p \) be the point in \([x x_1] \subset C \) with 
\( d(p, x) = d(p, y) = (c + b)/4 \). Then the geodesic bigon \( B = \{x, y\} \) given by the geodesics 
\([x x_1] \cup [x_1 y] \) and \([x_2 x] \cup [x_2 y] \) has \( \delta(B) \geq (c + b)/4 \), since \( d(p, x_2) = (c + a)/2 - (c + b)/4 + a = 
(c - b + 6a)/4 \geq (c + b)/4 \) (since \( b \leq 3a \)), and hence, \( d(p, [x_2y]) = d(p, \{x, y, x_2\}) = 
(c + b)/4 \). Since \( \text{diam} C_{a,b,c} = (c + b)/2 \), we have that \( \delta(C_{a,b,c}) \leq (c + b)/4 \) by Theorem 8. 
Therefore, in this case we have \( \delta(C_{a,b,c}) = \frac{c + b}{4} = \frac{c + \min\{b,3a\}}{4} \).

Assume now that \( b > 3a \). Let us consider a geodesic triangle \( T \); in order to compute 
\( \delta(C_{a,b,c}) \) without loss of generality we can assume that \( T \) is a simple cycle, by Corollary 10. 
If the closed curve given by \( T \) is \( C \cup A \), then \( \delta(T) \leq (c + a)/4 < (c + 3a)/4 \) and the first inequality is attained by taking \( T \) as a geodesic bigon. 
If the closed curve given by \( T \) is \( B \cup A \), then \( \delta(T) \leq (b + a)/4 < (c + 3a)/4 \) and the first inequality is attained by taking \( T \) as a geodesic bigon.

Assume that the closed curve given by \( T \) is \( C \cup B \). Let \( p \) be any point in \( T \). In order to compute \( \delta(T) \), by symmetry, without loss of generality we can assume that \( p \) belongs to a geodesic side \([xy]\) of \( T \) containing \( x_1 \) and that \( p \in C \). Also, we can assume that \( x \in C \) and 
\( y \in B \); then \( p \in [xx_1] \). If \( U := d(x, x_1) \) and \( V := d(y, x_1) \), then we have \( U \in [0, (c + a)/2] \)
and \( V \in [0, (b + a)/2] \) (in other case, \([xy]\) is not a geodesic). Since \([xy] = [xx_1] \cup [x_1y] \) is a 
geodesic, we also need \( U + V \leq (c + b)/2 \). Let \( \gamma_2, \gamma_3 \) be the other geodesics in \( T \). We denote 
by \( t \) the distance \( d(p, x) =: t \).

Define \( U_0 := (c-a)/2 \) and \( V_0 := (b+a)/2 \). Note that if \( U \leq U_0 \) and \( V \leq V_0 \), then \( U \) and \( V \) 
verify \( U + V \leq (c+b)/2 \) and \([xy]\) is a geodesic. Then \( d(p, \gamma_2 \cup \gamma_3) = \min \{t, U-t+a, U-t+V\} \)
and we have (since \( U \leq U_0 \), \( V \leq V_0 \) and \( a \leq V_0 \))

\[
\max_{p \in [xx_1]} d(p, \gamma_2 \cup \gamma_3) = \max_{t \in [0,U]} \min \{t, U-t+a, U-t+V\} \\
= \max_{t \in [0,U]} \min \{t, U_0-t+a, U_0-t+V_0\} \\
= \max_{t \in [0,U]} \min \{t, U_0-t+a\} = \frac{U_0+a}{2} = \frac{c+a}{4} < \frac{c+3a}{4}.
\]

Assume now that \( (c-a)/2 < U \leq (c+a)/2 \). Since we need \( U + V \leq (c+b)/2 \), then 
\( V \leq \min \left\{ \frac{b+a}{2}, \frac{c+b}{2} - U \right\} = \frac{c+b}{2} - U =: V_1 \),

and

\[
\max_{p \in [xx_1]} d(p, \gamma_2 \cup \gamma_3) = \max_{t \in [0,U]} \min \{t, U-t+a, U-t+V\} \\
\leq \max_{t \in [0,U]} \min \{t, U-t+a, U-t+V_1\} = \max_{t \in [0,U]} \min \left\{ t, U-t+a, \frac{c+b}{2} - t \right\}.
\]
Since

\[ U + a \leq \frac{c + a}{2} + a = \frac{c + 3a}{2} < \frac{c + b}{2}, \]

we deduce that

\[ \max_{p \in [xx_1]} d(p, \gamma_2 \cup \gamma_3) \leq \max_{t \in [0,t]} \min \left\{ t, U - t + a, \frac{c + b}{2} - t \right\} = \max_{t \in [0,t]} \min \left\{ t, U - t + a \right\} \]

\[ \leq \max_{t \in [0,t]} \min \left\{ t, \frac{c + 3a}{2} - t \right\} = \frac{c + 3a}{4}. \]

Since every inequality can be attained, we deduce \( \max_{p \in [xx_1]} d(p, \gamma_2 \cup \gamma_3) = \frac{c + 3a}{4} \). Therefore, we have \( \delta(C_{a,b,c}) = \frac{c + 3a}{4} = \frac{c + \min(b,3a)}{4} \).

Proposition 13. \( \delta(C_{a,b,c}) = \frac{1}{2} \text{ diam } C_{a,b,c} \) if and only if \( b \leq 3a \).

Proof. Using Theorem 12 and \( \text{diam } C_{a,b,c} = (c + b)/2 \), we have \( \delta(C_{a,b,c}) = \frac{1}{2} \text{ diam } C_{a,b,c} \) if and only if \( \frac{c + b}{4} = \frac{1}{2} \text{ diam } C_{a,b,c} = \delta(C_{a,b,c}) = \frac{c + \min(b,3a)}{4} \), and this holds if and only if \( b \leq 3a \).

\[ \square \]

3 Bounds on the hyperbolicity constant in a graph

A path \( \gamma \) between two points in a graph is called a bridge if the internal vertices of \( \gamma \) have degree two. In particular, any edge is a bridge, since it has no internal vertices.

Theorem 14. Assume that \( \gamma \) is a bridge in a graph \( G \) and \( \gamma' \) is a geodesic in the closure of \( G \setminus \gamma \) joining the same points as \( \gamma \). Then \( \max\{L(\gamma), L(\gamma')\} \leq 4\delta(G) \).

Proof. Let us denote by \( a \) and \( b \) the endpoints of \( \gamma \).

Assume first that \( \gamma \) is a geodesic joining \( a \) and \( b \); then \( L(\gamma) \leq L(\gamma') \). Let \( c \) be a point of \( \gamma' \) such that \( d_G(a,c) = d_G(b,c) = L(\gamma')/2 \); since \( \gamma' \) is a geodesic in the closure of \( G \setminus \gamma \), then \( \gamma' \) is the union of two geodesics (in \( G \)) \([ac]\) and \([cb]\). Let us consider the geodesic triangle \( T \) with sides \( \gamma, [ac], [cb] \). Let \( u \) be the midpoint of \([ac]\). Since \( \gamma \) is a bridge and \( \gamma' \) is a geodesic in the closure of \( G \setminus \gamma \), we have \( d_G(u, \{a,c\}) = d_G(u, \gamma \cup [cb]) \). Hence, \( \delta(T) \geq d_G(u, \{a,c\}) = L(\gamma')/4 \), and we conclude \( L(\gamma) \leq L(\gamma') \leq 4\delta(G) \).

Assume now that \( \gamma \) is not a geodesic; then \( \gamma' \) is a geodesic in \( G \) (since \( \gamma \) is a bridge), and \( L(\gamma') \leq L(\gamma) \). Using the previous argument, changing the role of \( \gamma \) and \( \gamma' \), we also deduce \( L(\gamma') \leq L(\gamma) \leq 4\delta(G) \).

A curve \( \gamma \) is a minimal closed geodesic if \( \gamma \) is a simple cycle such that for any two points of \( \gamma \), there exists a geodesic \( \gamma' \) joining them with \( \gamma' \subset \gamma \).

Remark 15. Every bridge is contained in a minimal closed geodesic.
Theorem 16. If \( G \) is any graph, then
\[
\delta(G) \geq \frac{1}{4} \sup \{ L(\gamma) : \gamma \text{ is a minimal closed geodesic} \}.\]

Proof. Consider any fixed minimal closed geodesic \( \gamma \). Let \( x, y \in \gamma \) such that \( d_G(x, y) = L(\gamma)/2 \). Then \( T = \{x, y\} \) is a bigon, with two geodesics \( \gamma_1, \gamma_2 \) verifying \( \gamma_1 \cup \gamma_2 = \gamma \). Let us consider \( u \in \gamma_1 \) with \( d_G(u, x) = d_G(u, y) = L(\gamma)/4 \). Since \( \gamma \) is a minimal closed geodesic, then \( d_G(u, \gamma_2) = d_G(u, \{x, y\}) = L(\gamma)/4 \), and \( \delta(G) \geq \delta(T) \geq L(\gamma)/4 \). This gives the result. \( \square \)

It is interesting to obtain inequalities involving the hyperbolicity constant and other important parameters of a graph. In this sense we obtain the following theorems.

Theorem 17. Let \( G \) be a graph with edges of length 1. If there exist a simple cycle \( g \) in \( G \) with length \( L(g) \geq 5 \) and a vertex \( w \in g \) with degree two, then \( \delta(G) \geq 5/4 \).

Proof. Let us denote by \( u, v \in g \) the two vertices which are the neighbors of \( w \), and by \( g_1 \) the subcurve of length 2 joining \( u \) and \( v \) and containing \( w \). Since the closure \( h \) of \( g \setminus g_1 \) is a curve in \( G \) joining \( u \) and \( v \) with \( L(h) \geq 3 \) and \( h \cap g_1 = \{u, v\} \), there exists a curve \( g_2 \) in \( G \) joining \( u \) and \( v \) with \( g_2 \cap g_1 = \{u, v\} \) and
\[
L(g_2) = \min \{ L(\sigma) : \sigma \text{ is a curve in } G \text{ joining } u \text{ and } v \text{ with } L(\sigma) \geq 3 \text{ and } \sigma \cap g_1 = \{u, v\} \}.
\]
Let \( z \) be the midpoint of \( g_2 \); it is clear that the two subarcs of \( g_2 \) joining \( z \) with \( u \) and \( v \) are geodesics by the minimizing property of \( g_2 \). Since \( w \) has degree two and \( u, v \) are the neighbors of \( w \), the two subarcs \( \gamma_1, \gamma_2 \) of \( \gamma := g_1 \cup g_2 \) joining \( z \) with \( w \) are geodesics.

Let us consider the bigon \( \{w, z\} \) with sides \( \gamma_1, \gamma_2 \), and the point \( p \in \gamma_1 \) at distance \( 5/4 \) from \( w \). Since \( L(\gamma_1) = L(\gamma_2) = L(\gamma)/2 \geq 5/2 \), we deduce \( d(p, \{w, z\}) \geq 5/4 \). If \( \sigma \) is any curve joining \( p \) and \( \gamma_2 \setminus \{w, z\} \), then \( L(\sigma \cap \gamma_1) \geq 1/4 \). Let \( q \in V(G) \) be the last point of \( \sigma \) in \( \gamma_1 \); then \( d(p, \gamma_2) = L(\sigma \cap \gamma_1) + d(q, \gamma_2) \geq 1/4 + 1 = 5/4 \). Then \( \delta(G) \geq 5/4 \). \( \square \)

Theorem 18. Let \( G \) be any graph with \( m \) edges. Then \( \delta(G) \leq \sum_{k=1}^{m} l_k/4 \), where \( l_k = L(e_k) \) for every edge \( e_k \in E(G) \). Moreover, \( \delta(G) = \sum_{k=1}^{m} l_k/4 \) if and only if \( G \) is isomorphic to \( C_m \).

Proof. It is not difficult to check the result for \( m = 1 \) (then the extremal graph is a vertex with a loop) and for \( m = 2 \) (in this case the extremal graph has two vertices and a double edge). Assume now that \( m \geq 3 \).

Let \( T \) be any fixed geodesic triangle, \( \gamma_1, \gamma_2, \gamma_3 \) be the geodesics joining the vertices of the triangle, and \( \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \) be the closed curve given by \( T \). In order to compute \( \delta(G) \), by Corollary 10, we can assume that \( \gamma \) is a simple cycle.

We have \( L(\gamma) \leq \sum_{k=1}^{m} l_k \), and hence \( L(\gamma_j) \leq \sum_{k=1}^{m} l_k/2 \), for every \( j \). If \( x \in \gamma_j =: [yz] \), then \( d(x, \{y, z\}) \leq L(\gamma_j)/2 \leq \sum_{k=1}^{m} l_k/4 \) and consequently \( \delta(T) \leq \sum_{k=1}^{m} l_k/4 \). Hence, \( \delta(G) \leq \sum_{k=1}^{m} l_k/4 \).

If \( \delta(G) = \sum_{k=1}^{m} l_k/4 \), then every inequality in the previous argument must be an equality. In particular, we have that \( L(\gamma) = \sum_{k=1}^{m} l_k \) and we deduce \( G = \gamma \). Therefore, we conclude that \( G \) is a simple cycle and, consequently, it is isomorphic to \( C_m \). \( \square \)
We deduce the following result for graphs with edges of length 1:

**Corollary 19.** Let \( G \) be any graph with \( m \) edges. If every edge has length 1, then \( \delta(G) \leq m/4 \). Moreover, \( \delta(G) = m/4 \) if and only if \( G \) is isometric to \( C_m \).

Given a graph \( G \), we say that a family of subgraphs \( \{G_n\}_n \) of \( G \) is a tree-decomposition of \( G \) if \( \bigcup_n G_n = G \), \( G_n \cap G_m \) is either a vertex or the empty set for each \( n \neq m \), and if the graph \( R \) is a tree, where \( V(R) = \{v_n\}_n \) and \( [v_n, v_m] \in E(R) \) if and only if \( G_n \cap G_m \neq \emptyset \).

We will need the following result (see [2]):

**Lemma 20.** Let \( G \) be a graph and \( \{G_n\}_n \) be a tree-decomposition of \( G \). Then \( \delta(G) = \sup_n \delta(G_n) \).

Furthermore, we have the following result.

**Theorem 21.** Let \( G \) be any graph with \( m \) edges. If every edge has length 1 and \( G \) is not isometric to \( C_m \), then \( \delta(G) \leq (m - 1)/4 \). Moreover, \( \delta(G) = (m - 1)/4 \) if and only if \( G \) is isometric to \( C_{m-1} \) with an edge \( e_0 \) attached, and we have either that \( e_0 \) is a loop or that the other vertex of \( e_0 \) has degree 1 or \( e_0 \) joins two different vertices of \( C_{m-1} \) at distance (in \( C_{m-1} \)) less or equal than 3.

**Proof.** Let \( T \) be a geodesic triangle, \( \gamma_1, \gamma_2, \gamma_3 \) be the geodesics joining the vertices of the triangle, and \( \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \) be the closed curve given by \( T \). In order to compute \( \delta(G) \), by Corollary 10, we can assume that \( \gamma \) is a simple cycle.

If \( L(\gamma) = m \), then \( \gamma = G \), and \( G \) is isometric to \( C_m \), which is a contradiction. Hence, \( L(\gamma) \leq m - 1 \) and \( L(\gamma_j) \leq (m - 1)/2 \) for every \( j \). If \( x \in \gamma_j =: [yz] \), then \( d(x, \{y, z\}) \leq L(\gamma_j)/2 \leq (m - 1)/4 \) and consequently \( \delta(T) \leq (m - 1)/4 \) and \( \delta(G) \leq (m - 1)/4 \).

If \( \delta(G) = (m - 1)/4 \), then every inequality in the previous argument must be an equality. Then we have that \( L(\gamma) = m - 1 \). Since \( \gamma \) is a simple cycle, we conclude that \( G \) is isometric to \( C_{m-1} \) with an edge \( e_0 \) attached.

A possibility is that \( e_0 \) is attached just in some vertex of \( C_{m-1} \). Then we have either that \( e_0 \) is a loop or that the other vertex of \( e_0 \) has degree 1. Both cases are possible, since \( \delta(G) = (m - 1)/4 \) by Lemma 20 (in both cases, \( \{\gamma, e_0\} \) is a tree-decomposition of \( G \)).

In other case, \( e_0 \) joins two different vertices of \( C_{m-1} \), and \( G \) is isometric to some \( C_{1,b,c} \), with \( b, c \in \mathbb{Z}^+ \), \( 1 + b + c = m \) and \( b \leq c \). Theorem 12 gives that \( \delta(C_{1,b,c}) = (c + \min\{b, 3\})/4 \). Hence, \( \delta(G) = (m - 1)/4 \) if and only if \( c + \min\{b, 3\} = m - 1 \), i.e., \( \min\{b, 3\} = b \) or \( b \leq 3 \). \( \square \)

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