On the exponent of convergence of negatively curved manifolds without Green’s function

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Abstract

In this paper we prove that for every complete $n$-dimensional Riemannian manifold without Green’s function and with its sectional curvatures satisfying $K \leq -1$, the exponent of convergence is greater than or equal to $n - 1$. Furthermore, we show that this inequality is sharp. This result is well known for manifolds with constant sectional curvatures $K = -1$.

Key words and phrases: Riemannian manifold; negative curvature; Green’s function; first eigenvalue; exponent of convergence.

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1 Introduction.

In this paper we prove same relations between some useful concepts of Potential Theory: the first eigenvalue of the Laplace-Beltrami operator, the existence of Green’s function and the exponent of convergence, for complete Riemannian $n$-manifolds ($n \geq 2$) with sectional curvatures $K \leq -1$. Note that the case of complete Riemannian $n$-manifolds with sectional curvatures $K \leq -k^2 < 0$ can be reduced to this one.

These concepts of Potential Theory are related with other interesting topics as, for instance, heat kernel, isoperimetric inequalities, conic limit set and geodesics in the manifold (see Section 2).

Our main result is Theorem 3.7, which states that for every complete $n$-dimensional Riemannian manifold without Green’s function and with sectional curvatures satisfying $K \leq -1$, the exponent of convergence is greater than or equal to $n - 1$. This inequality is sharp, as we show after the proof of this theorem. Theorem 3.7 also states that the first eigenvalue of every complete $n$-dimensional Riemannian manifold without Green’s function is 0 (without hypothesis on curvature). These results are well known for manifolds with constant sectional curvatures $K = -1$.

Note that to determine the first eigenvalue is not easy even if the manifold is negatively curved and its topology is trivial: if $M$ is a simply connected complete $n$-dimensional Riemannian manifold with sectional curvatures satisfying $K \leq -1$, we just know that $\lambda_1 \geq (n - 1)^2/4$ (see [15] and [20]).

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2 Some background on Potential Theory.

In this section we define the concepts on Potential Theory that we need. We start with the fundamental tone (the first eigenvalue of the Laplace-Beltrami operator).

**Definition 2.1.** Let $M$ be a Riemannian manifold and $\Omega \subseteq M$ be a domain in $M$. The fundamental tone $\lambda_1(\Omega)$ of $\Omega$ is defined in terms of the Rayleigh’s quotient as

$$
\lambda_1(\Omega) := \inf_{f \in C^\infty_c(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla f|^2}{\int_\Omega f^2}.
$$

It is well known that if $\Omega$ is a normal domain (a domain with compact closure and smooth non-empty boundary), the fundamental tone $\lambda_1(\Omega)$ is the lowest (first) eigenvalue of the positive Laplace-Beltrami operator $\Delta := -\text{div} \text{grad}$, with zero boundary value.

It is clear that in the definition of $\lambda_1(\Omega)$ we can replace the set $C^\infty_c(\Omega) \setminus \{0\}$ in the infimum by $W^{1,2}_0(\Omega)$, since $W^{1,2}_0(\Omega)$ is defined as the closure $C^\infty_c(\Omega)$ with the Sobolev norm.

We are mainly interested in $\lambda_1(M)$, i.e., in the case $\Omega = M$.

Eigenvalues of the Laplace-Beltrami operator play an important role in Riemannian geometry (see, e.g., [8]). In particular, the fundamental tone of a manifold is related with its exponent of convergence (see Theorem 3.5) and isoperimetric inequalities: a general result of Cheeger says that $\lambda_1 h^2 \geq 1/4$ for every complete Riemannian manifold with infinite volume, where $h$ is the smallest linear isoperimetric constant of the manifold; it turns out that negative curvature forces an inequality in the opposite direction (see [6]); hence, to have the linear isoperimetric inequality ($h < \infty$) in a negatively curved manifold is equivalent to $\lambda_1 > 0$. The linear isoperimetric inequality is closely related to the project of Ancona on the space of positive harmonic functions of Gromov-hyperbolic manifolds (see [1], [2] and [3]); in particular, Cao proves that for a large class of Gromov-hyperbolic manifolds (and graphs) the linear isoperimetric inequality implies that the Dirichlet problem at infinity for the Laplace-Beltrami operator is solvable (see [7]). Isoperimetric constants are also related with the geometry of ends and large time heat diffusion in Riemannian manifolds (see [9]).

We recall that a Green’s function in a complete Riemannian manifold $M$ is a positive fundamental solution of the Laplace-Beltrami operator on $M$. It is well known that a complete manifold has Green’s function if and only if there exists a non-constant positive superharmonic function (see, e.g., [21]). In terms of Brownian motion, a complete manifold has Green’s function if and only if the Brownian motion on the manifold is transient (i.e., the Brownian motion eventually escapes from any compact set with probability 1).

There is also another useful characterization of the existence of Green’s function:

**Definition 2.2.** The harmonic measure of the ideal boundary of a complete Riemannian manifold with respect to a normal domain $\Omega_0 \subseteq M$ is defined as follows. Let $\Omega$ be another normal domain containing $\overline{\Omega}_0$. We denote by $u_\Omega$ the harmonic function in $\overline{\Omega} \setminus \Omega_0$ which is 0 on $\partial \Omega_0$ and 1 on $\partial \Omega \setminus \partial \Omega_0$. The maximum principle implies that $u_\Omega$ decreases when $\Omega$ increases. Hence the harmonic limit function

$$
u_M(x) = \lim_{\Omega \to M} u_\Omega(x)
$$

is well defined in $M \setminus \Omega_0$. It is either identically 0 or positive and less than 1. In the first case we say that the harmonic measure of the ideal boundary vanishes; this is equivalent to

$$
\lim_{\Omega \to M} \int_{\Omega \setminus \Omega_0} |\nabla u_\Omega|^2 = 0,
$$

where $\Omega_0$ is a normal domain containing $\overline{\Omega}_0$.
since $\int_{\Omega \setminus \Omega_0} |\nabla u_\Omega|^2$ decreases when $\Omega$ increases: if $\Omega_0 \subset \subset \Omega \subset \subset \Omega'$ and we define $v := u_\Omega$ in $\Omega \setminus \Omega_0$ and $v := 1$ in $\Omega' \setminus \Omega$, then Dirichlet Principle's implies
\[ \int_{\Omega \setminus \Omega_0} |\nabla u_\Omega|^2 = \int_{\Omega' \setminus \Omega_0} |\nabla v|^2 \geq \int_{\Omega' \setminus \Omega_0} |\nabla u_\Omega'|^2. \]

It is easy to check that this definition does not depend on the choice of $\Omega_0$.

**Theorem 2.3.** ([21, p. 29]) A complete Riemannian manifold has not Green’s function if and only if the harmonic measure of the ideal boundary vanishes.

Green’s function is also related with other interesting topics as the heat kernel and isoperimetric inequalities: it is well known that Green’s function is the integral of the heat kernel of the manifold; Fernández proves in [10] that the existence of some kind of isoperimetric inequalities guarantees the existence of Green’s function for Riemannian manifolds; the results in [13] imply that the linear isoperimetric inequality guarantees the existence of Green’s function for negatively curved Riemannian surfaces.

**Definition 2.4.** Let $M$ be a complete $n$-dimensional Riemannian manifold and let us write $M = \tilde{M}/\Gamma$, where $\tilde{M}$ is a universal covering of $M$ and $\Gamma$ is a discrete group of isometries of $\tilde{M}$. The exponent of convergence $\delta(M)$ (see, e.g., [17, p. 21] for basic background) is defined as
\[ \delta(\Gamma) := \inf \left\{ t : \sum_{\gamma \in \Gamma} \exp \left( - t d(x, \gamma x) \right) < \infty, \quad \text{for some } x \in \tilde{M} \right\} \]
\[ = \inf \left\{ t : \sum_{[g] \in \Pi_1(p, M)} \exp \left( - t L([g]) \right) < \infty, \quad \text{for some } p \in M \right\}. \]

where
\[ L([g]) := \inf \left\{ L(\sigma) : \sigma \in [g] \right\}. \]

It is easy to check that if the series converges for some $x \in \tilde{M}$ (respectively, $p \in M$), then it converges for all $x \in \tilde{M}$ (respectively, $p \in M$).

In [19] it was proved that the exponent of convergence of a complete negatively curved manifold is equal to the Hausdorff dimension of its conic limit set (see the definition in [23]); it was previously proved in the case of constant curvature in [18] for surfaces of finite area, and in full generality in [11] and [5]. In particular, this fact implies that the set of bounded geodesics in the manifold (geodesics which are contained in a compact set, which depends of each geodesic) has Hausdorff dimension equal to the exponent of convergence (see [5], [11] and [19]). The exponent of convergence also plays an important role in the study of escaping geodesics in negatively curved surfaces (see [12] and [16]).

3 Potential Theory for manifolds with variable negative curvature.

By [14, Theorem 4.2] we have that on a Cartan-Hadamard manifold with sectional curvatures $K \leq -1$, the heat kernel is pointwise bounded from above by the heat kernel on the hyperbolic space of the same dimension. This fact and [14, Section 4.6] give the following result.

**Theorem 3.1.** There exists a constant $c_n$, which just depends on $n$, with the following property: for every $n$-dimensional Cartan-Hadamard manifold $M$ with sectional curvatures $K \leq -1$, the heat kernel satisfies
\[ p(x, y, t) \leq c_n \frac{(1 + r + t)^{(n-3)/2}(1 + r)}{t^{n/2}} \exp \left( - \frac{\lambda t - r^2}{4t} - \sqrt{\lambda} r \right), \]
where $\lambda := (n - 1)^2/4$ and $r := d(x, y)$, for every $x, y \in M$ and $t > 0$. 

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This inequality for the heat kernel allows to obtain an upper bound for the Green’s function as follows.

**Theorem 3.2.** For each \( n \geq 2 \) and \( r_0 > 0 \), there exists a positive constant \( C_1 \), which just depends on \( n \) and \( r_0 \), with the following property: for every \( n \)-dimensional Cartan-Hadamard manifold with sectional curvatures \( K \leq -1 \), the Green’s function satisfies

\[
g(x, y) \leq C_1 e^{-(n-1)r/2},
\]

for every \( x, y \in M \) with \( r := d(x, y) \geq r_0 \).

**Proof.** We just prove the case \( n \geq 3 \) (the case \( n = 2 \) is similar). Note first that

\[
f(r) := \frac{(1 + r + r^2)^{(n-3)/2}(1 + r)}{r^{n-2}} = \left( \frac{1 + r + r^2}{r^2} \right)^{(n-3)/2} 1 + r
\]
is a decreasing function on the interval \((0, \infty)\).

Using the well known formula \( g(x, y) = \int_0^\infty p(x, y, t) dt \) and Theorem 3.1, we obtain for every \( x, y \in M \) with \( r = d(x, y) \geq r_0 \) (making the change \( s = r^2/t \) in the first integral)

\[
g(x, y) \leq c_n \int_0^{r^2} \frac{(1 + r + t)^{(n-3)/2}(1 + r)}{t^{n/2}} \exp \left( -\frac{r^2}{4t} - \frac{(n-1)r}{2} \right) dt
\]

\[
+ c_n \int_{r^2}^\infty \frac{(1 + r + t)^{(n-3)/2}(1 + r)}{t^{n/2}} \exp \left( -\frac{r^2}{4t} - \frac{(n-1)r}{2} \right) dt
\]

\[
\leq c_n e^{-(n-1)r/2}(1 + r + r^2)^{(n-3)/2}(1 + r) \int_0^{r^2} \exp \left( -\frac{r^2}{4t} \right) dt
\]

\[
+ c_n e^{-(n-1)r/2} \int_{r^2}^\infty \frac{(1 + \sqrt{t} + t)^{(n-3)/2}(1 + \sqrt{t})}{t^{(n-2)/2}} e^{-\lambda t} \frac{dt}{t}
\]

\[
= c_n e^{-(n-1)r/2} f(r) \int_1^{\infty} e^{-s/4} s^{(n-4)/2} ds + c_n e^{-(n-1)r/2} \int_{r^2}^\infty f(\sqrt{t}) e^{-\lambda t} \frac{dt}{t}
\]

\[
\leq c_n f(r) e^{-(n-1)r/2} \int_1^{\infty} e^{-s/4} s^{(n-4)/2} ds + c_n f(r) e^{-(n-1)r/2} \int_{r^2}^\infty e^{-\lambda t} \frac{dt}{t}
\]

\[
\leq c_n f(r_0) \left( \int_1^{\infty} e^{-s/4} s^{(n-4)/2} ds + \int_{r_0^2}^{\infty} e^{-\lambda t} \frac{dt}{t} \right) e^{-(n-1)r/2} = C_1 e^{-(n-1)r/2},
\]

for \( r \geq r_0 \). \( \square \)

The upper bound obtained in Theorem 3.2 for the Green’s function in a Cartan-Hadamard manifold allows to obtain a criteria for existence of the Green’s function in a negatively curved manifold with arbitrary topology.

**Theorem 3.3.** Let \( M \) be a complete \( n \)-dimensional Riemannian manifold with sectional curvatures \( K \leq -1 \), and write \( \hat{M} = M/\Gamma \), where \( \hat{M} \) is a universal cover of \( M \) and \( \Gamma \) is a discrete group of isometries of \( M \). If the series

\[
\sum_{\gamma \in \Gamma} \exp \left( - (n-1)d(x_0, \gamma y_0)/2 \right)
\]

converges for some \( x_0, y_0 \in \hat{M} \), then there exists the Green’s function in \( M \).

**Proof.** It is clear, for the triangle inequality, that

\[
\sum_{\gamma \in \Gamma} \exp \left( - (n-1)d(x, \gamma y)/2 \right) < \infty
\]
for every $x, y \in \tilde{M}$. Let us denote by $g_{\tilde{M}}$ the Green’s function in $\tilde{M}$. If we denote by $\pi$ the universal cover map $\pi : \tilde{M} \longrightarrow M = \tilde{M}/\Gamma$, it is well known that if the series

$$\sum_{\gamma \in \Gamma} g_{\tilde{M}}(x, \gamma y)$$

converges for every $x, y \in \tilde{M}$, then there exists the Green’s function $g_M$ in $M$ and furthermore

$$g_M(\pi(x), \pi(y)) = \sum_{\gamma \in \Gamma} g_{\tilde{M}}(x, \gamma y).$$

Since $\Gamma$ is a discrete group of isometries of $\tilde{M}$, for each fixed choice of $x, y \in \tilde{M}$, we have that $\Gamma_{xy} := \{ \gamma \in \Gamma : d(x, \gamma y) < 1 \}$ is a finite set. Hence, applying Theorem 3.2,

$$g_M(\pi(x), \pi(y)) = \sum_{\gamma \in \Gamma_{xy}} g_{\tilde{M}}(x, \gamma y) + \sum_{\gamma \not\in \Gamma_{xy}} g_{\tilde{M}}(x, \gamma y) \leq \sum_{\gamma \in \Gamma_{xy}} g_{\tilde{M}}(x, \gamma y) + C_1 \sum_{\gamma \in \Gamma} e^{-(n-1)d(x, \gamma y)/2} < \infty.$$

As a consequence of the last theorem, we deduce the following result.

**Corollary 3.4.** For every complete $n$-dimensional Riemannian manifold with sectional curvatures $K \leq -1$ and without Green’s function, we have $\delta \geq (n-1)/2$.

A celebrated theorem of Patterson and Sullivan [22, p. 333] relates $\lambda_1(M)$ with $\delta(M)$ (if the sectional curvatures of $M$ are $K = -1$):

**Theorem 3.5.** For any complete $n$-dimensional Riemannian manifold $M$ with sectional curvatures $K = -1$, we have $\delta \in [0, n-1]$ and

$$\lambda_1 = \begin{cases} 
(n-1)^2/4, & \text{if } \delta \in [0, (n-1)/2], \\
\delta(n-1-\delta), & \text{if } \delta \in [(n-1)/2, n-1].
\end{cases}$$

Using the argument of Besson, Courtois and Gallot in [4, Lemma 5.3], the authors obtain in [16, Theorem 3.15] the following inequality, which is a version of the Patterson-Sullivan Theorem for the case of variable negative curvature.

**Theorem 3.6.** For every complete $n$-dimensional Riemannian manifold with sectional curvatures $K \leq -1$ we have $\lambda_1 \geq \delta(n-1-\delta)$.

Finally, we can prove our main result using Corollary 3.4 and Theorem 3.6.

**Theorem 3.7.** Every complete $n$-dimensional Riemannian manifold $M$ without Green’s function satisfies $\lambda_1 = 0$. Furthermore, if the sectional curvatures of $M$ satisfy $K \leq -1$, then $\delta \geq n-1$.

**Proof.** Theorem 2.3 gives that the harmonic measure of the ideal boundary of $M$ vanishes. Fix a normal domain $\Omega_0 \subset M$. For each $\varepsilon > 0$ we can find a normal domain $\Omega$ with compact closure, $\overline{\Omega_0} \subset \Omega$ and

$$\int_{\Omega \setminus \Omega_0} |\text{grad } u_\Omega|^2 < \varepsilon.$$

Let us define a function $f$ on $M$ as follows:

$$f := \begin{cases} 
1, & \text{on } \overline{\Omega_0}, \\
1 - u_\Omega, & \text{on } \Omega \setminus \Omega_0, \\
0, & \text{on } M \setminus \Omega.
\end{cases}$$
We have
\[ \lambda_1 \leq \frac{\int |\nabla f|^2}{\int f^2} \leq \frac{\int_{\Omega_0} |\nabla u_{\Omega_0}|^2}{\int_{\Omega_0} f^2} < \frac{\varepsilon}{Vol(\Omega_0)}. \]
Since \( \Omega_0 \) is fixed and \( \varepsilon \) is arbitrary, we deduce that \( \lambda_1 = 0 \).

Assume now that the sectional curvatures of \( M \) satisfy \( K \leq -1 \). Since Corollary 3.4 gives \( \delta \geq (n-1)/2 > 0 \) and Theorem 3.6 gives \( 0 \geq \delta(n - 1 - \delta) \), we obtain \( \delta \geq n - 1 \).

The inequality in Theorem 3.7 is sharp as the following examples show:

As a consequence of Theorems 3.5 and 3.7, if \( M \) is any complete \( n \)-dimensional Riemannian manifold without Green’s function and with sectional curvatures \( K = -1 \), then \( \delta = n - 1 \).

Furthermore, there exist manifolds \( M \) without Green’s function, with sectional curvatures satisfying \( K < -1 \), and \( \delta \) as close as we want to \( n - 1 \):

If \( M \) is any complete \( n \)-dimensional Riemannian manifold without Green’s function and with sectional curvatures satisfying \( K = -1 \), and \( g \) is its Riemannian metric, for each \( \varepsilon > 0 \) let us define \( g_\varepsilon := (1 + \varepsilon)^{-2}g \). It is easy to check that for \( (M, g_\varepsilon) \) we have \( K = -(1 + \varepsilon)^2 < -1 \) and \( \delta = (1 + \varepsilon)(n - 1) \).

References


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