Abstract. If $X$ is a geodesic metric space and $x_1, x_2, x_3$ are three points in $X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2], [x_2x_3]$ and $[x_3x_1]$ in $X$. The space $X$ is $\delta$-hyperbolic (in the Gromov sense) if any side of $T$ is contained in a $\delta$-neighborhood of the union of the two other sides, for every geodesic triangle $T$ in $X$. The study of hyperbolic graphs is an interesting topic since the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it. In this paper we obtain criteria which allow us to decide, for a large class of graphs, whether they are hyperbolic or not. We are especially interested in the planar graphs which are the “boundary” (the 1-skeleton) of a tessellation of the Euclidean plane. Furthermore, we prove that a graph obtained as the 1-skeleton of a general CW 2-complex is hyperbolic if and only if its dual graph is hyperbolic.

Keywords: Tessellation; planar graph; Gromov hyperbolicity; CW complex; dual graph.
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1. Introduction.

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [2, 3, 4, 6, 7, 8, 9, 16, 17, 18, 19, 20, 22, 23, 24, 26, 28, 29, 30, 33, 34, 35].

The theory of Gromov spaces was initially used to deal with finitely generated groups (see [11, 12] and the references therein), where it was demonstrated to have an enormous practical importance. This theory was applied principally to the study of automatic groups (see [25]), which play an important role in Science of Computation. Another important application of this spaces is secure transmission of information by the internet (see [16, 17, 18, 19, 20]). In particular, the hyperbolicity also plays an important role in the spread of viruses through the network (see [17, 18]). The hyperbolicity is also useful in the analysis of DNA data (see [6]).

In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring-Osgood $j$-metric is Gromov hyperbolic; and the Vuorinen $j$-metric is not Gromov hyperbolic except in the punctured space (see [13]). Gromov hyperbolicity of the quasi-hyperbolic and the Poincaré metrics is the subject of [1, 5, 14, 15, 30, 31, 34]. In particular, in [30, 34] it is proved the equivalence of the hyperbolicity of many Riemann surfaces (with their Poincaré metrics) and the hyperbolicity of a simple graph; hence, it is useful to know hyperbolicity criteria for graphs.

In our work on hyperbolic graphs we use the notations of [10]. Let $(X, d)$ be a metric space and let $\gamma : [a, b] \rightarrow X$ be a continuous function. We say that $\gamma$ is a geodesic if it is an isometry, i.e., $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t-s|$ for every $s, t \in [a, b]$, where $L$ denotes the length of a curve. We say that $X$ is a geodesic metric space if for every $x, y \in X$ there exists a geodesic joining $x$ and $y$; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient).
It is clear that every geodesic metric space is path-connected. If \( X \) is a graph, we use the notation \([u, v]\) for the edge of a graph joining the vertices \( u \) and \( v \).

In order to consider a graph \( G \) as a geodesic metric space, we must identify (by an isometry) any edge \([u, v] \in E(G)\) with a real interval with length \( l := L([u, v])\); therefore, any point in the interior of the edge \([u, v]\) is a point of \( G \). A connected graph \( G \) is naturally equipped with a distance induced by taking the shortest paths in \( G \), i.e., we consider \( G \) as a metric graph.

Throughout the paper we just deal with graphs which are connected and locally finite (i.e., each ball contains just a finite number of edges); we allow loops and multiple edges in the graphs; we also allow edges of arbitrary length. These conditions guarantee that the graph is a geodesic metric space (since we consider that every point in any edge of a graph \( G \) is a point of \( G \), whether it is a vertex of \( G \) or not).

If \( X \) is a geodesic metric space and \( J = \{J_1, J_2, \ldots, J_n\} \) is a polygon, with sides \( J_i \subseteq X \), we say that \( J \) is \( \delta \)-thin if for every \( x \in J_i \) we have that \( d(x, \cup_{j \neq i} J_j) \leq \delta \). We denote by \( \delta(J) \) the sharp thin constant of \( J \), i.e.,
\[
\delta(J) := \inf \{ \delta \geq 0 : J \text{ is } \delta\text{-thin} \}.
\]
If \( x_1, x_2, x_3 \in X \), a geodesic triangle \( T = \{x_1, x_2, x_3\} \) is the union of the three geodesics \([x_1, x_2], [x_2, x_3]\) and \([x_3, x_1]\). The space \( X \) is \( \delta \)-hyperbolic (or satisfies the Rips condition with constant \( \delta \)) if every geodesic triangle in \( X \) is \( \delta \)-thin. We denote by \( \delta(X) \) the sharp hyperbolicity constant of \( X \), i.e.,
\[
\delta(X) := \sup \{ \delta(T) : T \text{ is a geodesic triangle in } X \}.
\]
We say that \( X \) is hyperbolic if \( X \) is \( \delta \)-hyperbolic for some \( \delta \geq 0 \). If \( X \) is hyperbolic, then \( \delta(X) = \inf \{ \delta \geq 0 : X \text{ is } \delta\text{-hyperbolic} \} \). Note that if \( X \) is hyperbolic, then every geodesic polygon with \( n \) sides (\( n \geq 3 \)) in \( X \) is \((n - 2)\delta(X)\)-thin.

The following are interesting examples of hyperbolic spaces. The real line \( \mathbb{R} \) is \( 0 \)-hyperbolic: in fact, any point of a geodesic triangle in the real line belongs to two sides of the triangle simultaneously, and therefore we can conclude that \( \mathbb{R} \) is \( 0 \)-hyperbolic. The Euclidean plane \( \mathbb{R}^2 \) is not hyperbolic: it is clear that equilateral triangles can be drawn with arbitrarily large diameter, so that \( \mathbb{R}^2 \) with the Euclidean metric is not hyperbolic. This argument can be generalized in a similar way to higher dimensions: a normed vector space \( E \) is hyperbolic if and only if \( \dim E = 1 \). Every metric tree with arbitrary length edges is \( 0 \)-hyperbolic: in fact, all points of a geodesic triangle in a tree belong simultaneously to two sides of the triangle. Every bounded metric space \( X \) is \((\operatorname{diam} X)\)-hyperbolic. Every simply connected complete Riemannian manifold with sectional curvature verifying \( K \leq -c^2 \), for some positive constant \( c \), is hyperbolic. We refer to [10] for more background and further results.

We would like to point out that deciding whether or not a space is hyperbolic is usually extraordinarily difficult: Note that, first of all, we have to consider an arbitrary geodesic triangle \( T \), and calculate the minimum distance from an arbitrary point \( P \) of \( T \) to the union of the other two sides of the triangle to which \( P \) does not belong to. And then we have to take the supremum over all the possible choices for \( P \) and then over all the possible choices for \( T \). Without disregarding the difficulty of solving this minimax problem, notice that in general the main obstacle is that we do not know the location of geodesics in the space. Therefore, it is interesting to obtain inequalities relating the hyperbolicity constant to other parameters of graphs.

Since to obtain a characterization of hyperbolic graphs is, perhaps, too ambitious, it seems reasonable to study this problem for a particular class of graphs. In [8], [24] and [26] the authors study line graphs, product graphs and cubic graphs, respectively. One of the aims of this paper is to obtain information about either the hyperbolicity or the non-hyperbolicity of a wide class of planar graphs: the graphs which are the “boundary” (the 1-skeleton) of a tessellation of the Euclidean plane. The edges of such a tessellation graph are just rectifiable paths in \( \mathbb{R}^2 \) and have the length induced by the metric in \( \mathbb{R}^2 \) (they may or may not be geodesics in \( \mathbb{R}^2 \)).

In fact, in Section 3 we provide several criteria in order to conclude that many tessellation graphs of the Euclidean plane \( \mathbb{R}^2 \) are non-hyperbolic. One can think that the tessellation graphs of the Euclidean plane \( \mathbb{R}^2 \) are always non-hyperbolic, since the plane is non-hyperbolic (and then the theory would be trivial); however, there exists a hyperbolic tessellation graph of \( \mathbb{R}^2 \) (see [29]).

These tessellation graphs are the 1-skeleton (i.e., the set of 1-cells and 0-cells) of a CW 2-complex contained in the Euclidean plane. In Section 4 we deal with a wider class of graphs: the graphs which are the 1-skeleton
of an abstract CW 2-complex (not necessarily contained in $\mathbb{R}^2$ or in some Riemannian surface). In fact, we prove that a graph obtained as the 1-skeleton of a CW 2-complex is hyperbolic if and only if its dual graph is hyperbolic, under some reasonable hypotheses (see Theorem 4.4). This result improves [29, Theorem 4.1] about a particular kind of CW 2-complexes: the ones that are a tessellation of some complete Riemannian surface without boundary (in this special case every edge belongs exactly to two faces). Furthermore, Section 4 contains Theorem 4.10, which is a stronger version of Theorem 4.4.

**Notations.** If $X$ is a geodesic metric space, then by $d_X$, $L_X$ and $B_X$ we shall denote, respectively, the distance, the length and the balls in the metric of $X$.

Throughout the paper, we say that an inequality holds **quantitatively** if it holds with a constant depending only on the constants in the assumptions.

2. Background and previous results.

Let $(X,d_X)$ and $(Y,d_Y)$ be two metric spaces. A map $f : X \rightarrow Y$ is said to be an $(\alpha, \beta)$-**quasi-isometric embedding**, with constants $\alpha \geq 1$, $\beta \geq 0$ if we have for every $x, y \in X$:

$$\alpha^{-1} d_X(x,y) - \beta \leq d_Y(f(x),f(y)) \leq \alpha d_X(x,y) + \beta.$$  

We say that $f$ is $\varepsilon$-full if for each $y \in Y$ there exists $x \in X$ with $d_Y(f(x),y) \leq \varepsilon$.

A map $f : X \rightarrow Y$ is said to be a **quasi-isometry**, if there exist constants $\alpha \geq 1$, $\beta, \varepsilon \geq 0$ such that $f$ is a $\varepsilon$-full $(\alpha, \beta)$-quasi-isometric embedding.

Two metric spaces $X$ and $Y$ are **quasi-isometric** if there exists a quasi-isometry $f : X \rightarrow Y$.

A fundamental property of hyperbolic spaces is the following (see, e.g., [10, p.88]):

**Theorem 2.1** (Invariance of hyperbolicity). Let $f : X \rightarrow Y$ be an $(\alpha, \beta)$-quasi-isometric embedding between two geodesic metric spaces. If $Y$ is $\delta$-hyperbolic, then $X$ is $\delta'$-hyperbolic, quantitatively.

Besides, if $f$ is $\varepsilon$-full for some $\varepsilon \geq 0$ (a quasi-isometry), then $X$ is $\delta'$-hyperbolic if and only if $Y$ is $\delta$-hyperbolic, quantitatively.

We will need the following result (see [29, Theorem 3.1]):

**Theorem 2.2.** Let $S$ be a Riemannian surface with curvature satisfying $K \geq -k^2$ for some constant $k$. Suppose that a graph $G$ is the 1-skeleton of a tessellation of $S$ with tiles $\{F_n\}$ such that there exist sets $\Lambda_1, \Lambda_2$ which are a partition of the sets of indices $n$, and positive constants $c_1, c_2$, verifying the following properties: $\text{diam}_S \partial F_n \leq c_1$, $\text{diam}_S F_n \leq c_1$ and $\text{diam}_S F_n \geq c_2$ for every $n \in \Lambda_1$, and $d_{\partial F_n}(x,y) \leq c_1 d_S(x,y)$ for every $x, y \in \partial F_n$ and for every $n \in \Lambda_2$. If $S$ is hyperbolic, then $G$ is hyperbolic, quantitatively.

Furthermore, if $\text{diam}_S F_n \leq c_1$ for every $n \in \Lambda_2$, then $S$ is hyperbolic if and only if $G$ is hyperbolic, quantitatively.

We will also need the following results (see [33, Theorem 11 and Lemma 5]).

**Theorem 2.3.** The following graphs with edges of length 1 have these precise values of $\delta$:

- The path graphs verify $\delta(P_n) = 0$ for every $n \geq 1$.
- The cycle graphs verify $\delta(C_n) = n/4$ for every $n \geq 3$.
- The complete graphs verify $\delta(K_1) = \delta(K_2) = 0$, $\delta(K_3) = 3/4$, $\delta(K_n) = 1$ for every $n \geq 4$.
- The complete bipartite graphs verify $\delta(K_{1,1}) = \delta(K_{1,2}) = \delta(K_{2,1}) = 0$, $\delta(K_{m,n}) = 1$ for every $m, n \geq 2$.
- The Petersen graph $P$ verifies $\delta(P) = 3/2$.
- The wheel graph with $n$ vertices $W_n$ verifies $\delta(W_4) = \delta(W_5) = 1$, $\delta(W_n) = 3/2$ for every $7 \leq n \leq 10$, and $\delta(W_n) = 5/4$ for $n = 6$ and for every $n \geq 11$.

We say that a subgraph $\Gamma$ of $G$ is isometric if $d_{\Gamma}(x,y) = d_G(x,y)$ for every $x, y \in \Gamma$. 
Lemma 2.4. If $\Gamma$ is an isometric subgraph of $G$, then $\delta(\Gamma) \leq \delta(G)$.

3. Hyperbolicity of planar graphs.

We obtain in this section additional results on the hyperbolicity of tessellation graphs of $\mathbb{R}^2$. The main results in this section are Theorems 3.1 and 3.8, since they are the key tools in order to prove the other results.

We denote by $\text{int}(F)$ the topological interior of the set $F$.

Theorem 3.1. Suppose that a graph $G$ is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}$. Denote by $c_n$ the shortest cycle in $G$ homotopic to $\partial F_n$ in $\mathbb{R}^2 \setminus \text{int}(F_n)$. If $\sup_n L(c_n) = \infty$, then $G$ is not hyperbolic.

Proof. First, we prove that $c_n$ is an isometric subgraph of $G$ for every $n$. Seeking for a contradiction assume that there exists $n$ such that $c_n$ is not an isometric subgraph of $G$. Then there exist $x, y \in c_n$ and a curve $\gamma$ in $G$ joining them with $L(\gamma) < d_{c_n}(x, y)$; therefore, if $g_1, g_2$ are the two curves joining $x$ and $y$ with $g_1 \cup g_2 = c_n$, we have $L(\gamma) < \min \{L(g_1), L(g_2)\}$. Since $c_n$ is homotopic to $\partial F_n$ in $\mathbb{R}^2 \setminus \text{int}(F_n)$, we have that either $g_1 \cup g_2$ or $g_1 \cup g_2$ is homotopic to $\partial F_n$ in $\mathbb{R}^2 \setminus \text{int}(F_n)$. Since $\max \{L(g_1), L(g_2)\} < L(c_n)$, we have the contradiction we were aiming for, and we conclude that $c_n$ is an isometric subgraph of $G$ for every $n$.

Let us fix $n$. If $\gamma_1, \gamma_2$ are two curves with $\gamma_1 \cup \gamma_2 = c_n$ and $L(\gamma_1) = L(\gamma_2) = L(c_n)/2$, then $B = \{\gamma_1, \gamma_2\}$ is a geodesic bigon (a geodesic triangle such that two of its vertices are the same point) in $c_n$. If we denote by $z$ the midpoint of $\gamma_1$, then $B \geq d_{c_n}(z, \gamma_2) = L(c_n)/4$. Lemma 2.4 gives $\delta(G) \geq \delta(c_n) \geq \delta(B) \geq L(c_n)/4$ for every $n$, and we deduce that $G$ is not hyperbolic.

Corollary 3.2. Suppose that a graph $G$ is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}$. Let us assume that there exists a subsequence of tiles $\{F_{n_k}\}_k$ such that they are all convex polygons and, besides, $\sup_k L(\partial F_{n_k}) = \infty$. Then $G$ is not hyperbolic.

Proof. Since each $F_{n_k}$ is a convex polygon, we have $c_{n_k} = \partial F_{n_k}$, where $c_{n_k}$ are the shortest cycles mentioned in Theorem 3.1. Applying that Theorem, the conclusion is straightforward.

Corollary 3.3. Suppose that a graph $G$ is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}$. If there exists a sequence of balls $\{B_n\}$ with radius $r_n$ such that $B_n \subseteq F_n$ for every $n$ and $\sup_n r_n = \infty$, then $G$ is not hyperbolic.

Proof. Let us consider the cycles $c_n$ as in Theorem 3.1. For each $n$ it is obvious that $L(c_n) \geq L(\partial B_n) = 2\pi r_n$. Therefore, $\sup_n L(c_n) = \infty$ and Theorem 3.1 gives the conclusion.

Theorem 3.4. Suppose that a graph $G$ is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with tiles $\{F_n\}$. Assume that every tile $F_n$ can be obtained from a finite set of tiles $\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_m$ by means of translations, rotations and dilations. Then, $G$ is not hyperbolic.

Proof. Since every tile $F_n$ can be obtained from a finite set of tiles $\hat{F}_1, \hat{F}_2, \ldots, \hat{F}_m$ by means of translations, rotations and dilations, there exists a positive constant $c_1$ such that $d_{\partial F_n}(x, y) \leq c_1 d_{\hat{F}_n}(x, y)$ for every $x, y \in \partial F_n$ and for every $n$.

Let us consider the cycles $c_n$ as in Theorem 3.1. If $\sup_n L(\partial F_n) = \infty$, then $\sup_n L(c_n) = \infty$ and Theorem 3.1 gives the result. Assume now that $\sup_n L(\partial F_n) < \infty$. Since

$$\sup_n \text{diam}_{\hat{F}_n} F_n = \sup_n \text{diam}_{\hat{F}_n} \partial F_n \leq \sup_n \text{diam}_{\partial F_n} \partial F_n \leq \frac{1}{2} \sup_n L(\partial F_n) < \infty,$$

and $\mathbb{R}^2$ is not hyperbolic, Theorem 2.2 (with $\Lambda_1 = \emptyset$) gives that $G$ is not hyperbolic.

Theorem 3.5. Suppose that a graph $G$ is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with convex tiles $\{F_n\}$. If $\inf_n A(F_n) > 0$, then $G$ is not hyperbolic.
Proof. If \( \sup_n L(\partial F_n) = \infty \), then Theorem 3.1 gives the result. Assume now that \( \sup_n L(\partial F_n) = c_1 < \infty \). Since
\[
\text{diam}_{\mathbb{R}^2} F_n = \text{diam}_{\mathbb{R}^2} \partial F_n \leq \text{diam}_{G} \partial F_n \leq \frac{1}{2} L(\partial F_n) \leq \frac{c_1}{2},
\]
\( \inf_n A(F_n) > 0 \) and \( \mathbb{R}^2 \) is not hyperbolic, Theorem 2.2 (with \( A_2 = \emptyset \)) gives that \( G \) is not hyperbolic. \( \square \)

**Theorem 3.6.** Suppose that a graph \( G \) is the 1-skeleton of a tessellation of \( \mathbb{R}^2 \) with tiles \( \{F_n\} \). If every tile \( F_n \) is a regular polygon, then \( G \) is not hyperbolic.

Proof. If \( \sup_n L(\partial F_n) = \infty \), then Theorem 3.1 gives the result. Assume now that \( \sup_n L(\partial F_n) < \infty \).

Let us fix \( n \) and \( x, y \in \partial F_n \). We define \( k_n \) as the number of edges in \( F_n, \omega_n \) as the inradius of \( F_n, \Omega_n \) as the circumcircle of \( F_n, R_n \) as the circumradius of \( F_n \) and \( O_n \) as the circumcenter of \( F_n \). Also, we define \( C_n \) as the circumference with center on \( O_n \) and radius \( d_{\mathbb{R}^2}(O_n, x) \) for \( x \in \partial F_n \).

If \( x, y \) belong to the same edge, then \( d_{\partial F_n}(x, y) = d_{\mathbb{R}^2}(x, y) \). In other case, there exists a non-empty set \( \{V_i\}_{i=0}^n \subset V(G) \cap \partial F_n \) such that \( [xV_0] \cup ([xV_i(xV_{i+1})] \cup [V_0y] \) is a geodesic in \( G \) joining \( x \) and \( y \). It is clear that \( d_{\partial F_n}(V_i, V_{i+1}) \leq d_{\mathbb{R}^2}(V_i, V_{i+1}) \) for all \( i = 0, \ldots, r - 1 \).

Besides, if \( x' \in \partial a \) verifies \( d_{\partial F_n}(a, x) + d_{\partial F_n}(x, x') = R_n \) then, by the cosine rule we obtain \( d_{\partial F_n}(x, y) \leq d_{\mathbb{R}^2}(x', V_0) \) since \( R_n \cos \frac{x}{k_n} = r_n \leq d_{\mathbb{R}^2}(a, x) \leq d_{\mathbb{R}^2}(x', V_0) = R_0 \) and \( \leq \angle xO_nV_0 \leq \frac{\pi}{3} \):
\[
\begin{align*}
d_{\partial F_n}(V_0, x') &= R_n^2 + 2R_nR_n \cos \beta, & \text{with } \beta = \angle xO_nV_0 \leq \frac{\pi}{3}, \\
d_{\partial F_n}(V_0, x) &= R_n^2 + \alpha^2 R_n^2 - 2R_n \alpha \cos \beta, & \text{with } \cos \frac{\pi}{k_n} \leq \alpha \leq 1, \\
d_{\partial F_n}(V_0, x') - d_{\partial F_n}(V_0, x) &= R_n^2(1 - 2 \cos \beta - \alpha^2 + 2 \alpha \cos \beta) \geq 0.
\end{align*}
\]

So, \( d_{\partial F_n}(x, V_0) \leq d_{\mathbb{R}^2}(x', V_0) \). Obviously, we can obtain a similar result for \( y \). Then, we obtain \( d_{\partial F_n}(x, y) \leq d_{\mathbb{R}^2}(x', y') \). Without loss of generality we can assume that \( r_n, x = d_{\mathbb{R}^2}(O_n, x) \leq d_{\mathbb{R}^2}(O_n, y) \). Therefore, \( d_{\partial F_n}(x, y) \leq \frac{R_n}{r_n} d_{\partial F_n}(x, y') \) where \( y' = C_n(x, y) \). Since \( \frac{r_n}{R_n} \cos \frac{x}{k_n} \geq \frac{1}{2} \) we have that \( d_{\partial F_n}(x, y) \leq 2d_{\partial F_n}(x, y') \). It is known that \( d_{\mathbb{R}^2}(x, y') \leq \frac{\pi}{2} d_{\mathbb{R}^2}(x, y') \), and then \( d_{\mathbb{R}^2}(x, y') \leq \pi d_{\mathbb{R}^2}(x, y') \). By the cosine rule in the triangles \( \angle xO_ny' \) and \( \angle xO_ny \) at vertex \( O_n \), since \( d_{\mathbb{R}^2}(O_n, y') \leq d_{\mathbb{R}^2}(O_n, y) \), we obtain \( d_{\mathbb{R}^2}(x, y') \leq d_{\mathbb{R}^2}(x, y') \):
\[
\begin{align*}
d_{\mathbb{R}^2}(x, y') &= a^2 + 2 \alpha - 2 \alpha \cos(\angle xO_ny), & \text{with } a = d_{\mathbb{R}^2}(O_n, y') = d_{\mathbb{R}^2}(O_n, x), \\
d_{\mathbb{R}^2}(x, y) &= a^2 + (a + \alpha)^2 - 2(a + \alpha) \cos(\angle xO_ny), & \text{with } a = d_{\mathbb{R}^2}(y', y); \\
d_{\mathbb{R}^2}(x, y) - d_{\mathbb{R}^2}(x, y') &= a^2 + 2 \alpha(1 - \cos(\angle xO_ny)).
\end{align*}
\]

Finally, we obtain \( d_{\partial F_n}(x, y) \leq \pi d_{\mathbb{R}^2}(x, y) \) for all \( x, y \in \partial F_n \). Then, Theorem 2.2 (with \( A_1 = \emptyset \)) gives that \( G \) is not hyperbolic, since \( \mathbb{R}^2 \) is not hyperbolic. \( \square \)

In order to prove our next theorem, we will need the following well-known (and non-trivial) result.

**Lemma 3.7.** Given any open convex set \( C \subset \mathbb{R}^2 \) and any curve \( g \subset \mathbb{R}^2 \setminus C \) joining two points \( x, y \in \partial C \), there exists a curve \( \gamma \subset \partial C \) joining \( x \) and \( y \) with \( L(\gamma) \leq L(g) \).

**Theorem 3.8.** Suppose that a graph \( G \) is the 1-skeleton of a tessellation of \( \mathbb{R}^2 \) with convex tiles \( \{F_n\} \). Let us assume that there exist balls \( B_n \subset F_n \) with radius \( r_n \) such that \( L(\partial F_n) \leq c_1 r_n \) for some positive constant \( c_1 \) and every \( n \). Then \( G \) is not hyperbolic.

Proof. By Corollary 3.2, we can assume that there exists a constant \( k_1 \) with \( L(\partial F_n) \leq 2k_1 \) for every \( n \). Note that \( \text{diam}_{\mathbb{R}^2} F_n \leq \frac{1}{2} L(\partial F_n) =: R_n \leq k_1 \) for every \( n \). If we denote by \( B_n^* \) the closed ball with the same center
as $B_n$ and radius $R_n$, then we have $B_n \subset F_n \subset B^*_n$ and

\[(3.1) \quad \frac{R_n}{r_n} \leq \frac{\frac{1}{2}L(\partial F_n)}{c_1L(\partial F_n)} = \frac{c_1}{2}.
\]

Let us consider $z, w \in \partial F_n$. We want to show that there exists a constant $c$, which just depends on $c_1$, such that $d_{F_n}(z, w) \leq c d_{g_2}(z, w)$. On the one hand, if $d_{g_2}(z, w) \geq r_n$, then

\[d_{F_n}(z, w) \leq \frac{1}{2}L(\partial F_n) \leq \frac{1}{2}L(\partial F_n) \frac{d_{g_2}(z, w)}{r_n} \leq \frac{c_1}{2}d_{g_2}(z, w).
\]

On the other hand, let us consider the case $d_{g_2}(z, w) < r_n$. Without loss of generality we can assume that 0 is the center of $B_n$ and $B^*_n$. Since $d_{g_2}(z, w) < r_n$, we have that $|\arg z - \arg w| < \pi/2$; let us consider the straight lines $S_z$ and $S_w$ containing the edges in $\partial F_n$ which contain $z$ and $w$, respectively; let $\zeta := S_z \cap S_w$ (note that if the edges in $\partial F_n$ which contain $z$ and $w$ are not adjacent, then $[\zeta] \not\subset G$ and $[\zeta] \not\subset G$).

Since $F_n$ is convex, Lemma 3.7 gives that

\[d_{F_n}(z, w) \leq \frac{L([\zeta] \cup [\zeta])}{d_{g_2}(z, w)}.
\]

In order to bound $L([\zeta] \cup [\zeta])/d_{g_2}(z, w)$, we are going to find the maximum

\[(3.2) \quad \max \left\{ \frac{L([u\zeta] \cup [v\zeta])}{d_{g_2}(u, v)} : u \in [\zeta], v \in [\zeta] \right\}.
\]

Let us denote by $\alpha$ the angle at $\zeta$ of $[\zeta]$ and $[\zeta]$ ($0 < \alpha \leq \pi$). By (3.1) and $d_{g_2}(z, w) < r_n$, we deduce that $\alpha \geq \alpha_0$, with $\alpha_0$ a constant which just depends on $c_1$. It is not difficult to check that the maximum in (3.2) is attained if $u \in [\zeta]$ and $v \in [\zeta]$ with $d_{g_2}(u, v) = d_{g_2}(v, \zeta)$, and that it is equal to $1/\sin(\alpha/2)$. Hence, we can conclude that

\[d_{F_n}(z, w) \leq \frac{L([\zeta] \cup [\zeta])}{d_{g_2}(z, w)} \leq \max \left\{ \frac{L([u\zeta] \cup [v\zeta])}{d_{g_2}(u, v)} : u \in [\zeta], v \in [\zeta] \right\} = \frac{1}{\sin(\alpha/2)} \leq \frac{1}{\sin(\alpha_0/2)}.
\]

Therefore,

\[d_{F_n}(z, w) \leq \max \left\{ \frac{c_1}{2}, \frac{1}{\sin(\alpha_0/2)} \right\} d_{g_2}(z, w),
\]

for every $z, w \in \partial F_n$ and for every $n$. Now, since $\text{diam}_{g_2} F_n \leq k_1$ for every $n$, Theorem 2.2 (with $A_1 = \emptyset$) finishes the proof. \[\square\]

Suppose that a graph $G$ is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with convex tiles $\{F_n\}$. Let us define

\[l_n := \min \{L_G(e) : e \in E(G), e \subset \partial F_n\},
\]

\[L_n := \max \{L_G(e) : e \in E(G), e \subset \partial F_n\},
\]

\[\alpha_n := \min \{\text{interior angles at the vertices in } \partial F_n\},
\]

\[N_n := \text{card} \{e \in E(G) : e \subset \partial F_n\}.
\]

**Theorem 3.9.** Suppose that a graph $G$ is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with convex tiles $\{F_n\}$. Let us assume that $L(\partial F_n) \leq c_1 l_n$ and $\alpha_n \geq c_2$ for some positive constants $c_1, c_2$ and for every $n$. Then $G$ is not hyperbolic.

**Proof.** For each fixed $n$, let us consider two adjacent edges $e_1^n, e_2^n$ contained in $\partial F_n$ such that $\alpha_n$ at the point $V_n = e_1^n \cap e_2^n$. We have that $N_n \leq c_1$, since $L(\partial F_n) \leq c_1 l_n$. So, we have that $\alpha_n \leq \frac{N_n}{N_n-2} \pi$, since $G$ is a convex tessellation ($F_n$ is a polygon). Let us consider $u_1 \in e_1^n$ and $u_2 \in e_2^n$ such that $d_{g}(u_1, V_n) = d_{g}(V_n, u_2) = l_n$. If $A_n$ is the Euclidean convex hull in $\mathbb{R}^2$ of $\{u_1, V_n, u_2\}$ and $B_n$ the in-circle of $A_n$ with radius $r_n$, then $L(\partial A_n) \leq 6r_n \tan \frac{\alpha}{2}$ with $\alpha = \min\{c_2, \frac{\pi}{c_1}\}$. The previous result is obvious since $\min \{\text{interior angles at the vertices in } \partial A_n\} \geq \min\{c_2, \frac{\pi}{c_1}\}$ and the monotony of the $\tan$ function on
Suppose that a graph $G$ is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with convex tiles $\{F_n\}$. Let us assume that $L_n \leq c_1 l_n$, $N_n \leq c_1$ and $\alpha_n \geq c_2$ for some positive constants $c_1, c_2$ and for every $n$. Then $G$ is not hyperbolic.

We obtain the following results from Corollary 3.10.

**Corollary 3.11.** Suppose that a graph $G$ is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with triangular tiles $\{F_n\}$. Let us assume that $\alpha_n \geq c_2$ for some positive constant $c_2$ and for every $n$. Then $G$ is not hyperbolic.

**Corollary 3.12.** Suppose that a graph $G$ is the 1-skeleton of a tessellation of $\mathbb{R}^2$ with rectangular tiles $\{F_n\}$. Let us assume that $L_n \leq c_1 l_n$ for some positive constant $c_1$ and for every $n$. Then $G$ is not hyperbolic.

**Open problem.** At the light of these results we conjecture that every tessellation graph of $\mathbb{R}^2$ with convex tiles is non-hyperbolic. The proof of this conjecture would use different arguments, since some tessellation graphs of $\mathbb{R}^2$ with convex tiles are not quasi-isometric to $\mathbb{R}^2$.

### 4. Hyperbolicity of dual graphs.

In this section we get results for a class of geodesic metric spaces wider than the tessellation graphs of the plane. First of all we give the precise definition of CW complex.

**Definition 4.1.** Let $D^n$ be the closed unit ball in $\mathbb{R}^n$. An $n$-cell ($n \geq 1$) is a space homeomorphic to the open $n$-ball $\text{int}(D^n)$; a 0-cell is a single point. A cell is a space which is an $n$-cell for some $n \geq 0$.

Note that $\text{int}(D^n)$ and $\text{int}(D^m)$ are homeomorphic if and only if $m = n$. Thus we can talk about the dimension of a cell. An $n$-cell will be said to have dimension $n$.

**Definition 4.2.** A cell-decomposition of a space $X$ is a family $\xi = \{e_\alpha| \alpha \in I\}$ of subspaces of $X$ such that each $e_\alpha$ is a cell and

$$X = \bigcup_{\alpha \in I} e_\alpha$$

(disjoint union of sets). The $n$-skeleton of $X$ is the subspace $X^n = \bigcup_{\alpha \in I : \dim(e_\alpha) \leq n} e_\alpha$.

**Definition 4.3.** A pair $(X, \xi)$ consisting of a Hausdorff space $X$ and a cell-decomposition $\xi$ of $X$ is called a CW-complex if the following axioms are satisfied:

**Axiom 1:** ("Characteristic Maps") For each $n$-cell $e \in \xi$ ($n \geq 1$) there is a map $\Phi_e : D^n \rightarrow X$ restricting to a homeomorphism $\Phi_e|\text{int}(D^n) : \text{int}(D^n) \rightarrow e$ and taking $S^{n-1}$ into $X^{n-1}$.

**Axiom 2:** ("Closure Finiteness") For any cell $e \in \xi$ the closure $\overline{e}$ intersects only a finite number of other cells in $\xi$.

**Axiom 3:** ("Weak Topology") A subset $A \subseteq X$ is closed if and only if $A \cap \overline{e}$ is closed in $X$ for each $e \in \xi$.

We consider in this section a very large class of graphs which contains the tessellation graphs of complete Riemannian surfaces (with or without boundary): the set of every graph $G$ which is the 1-skeleton (the set of 0-cells and 1-cells) of some connected CW 2-complex. The dual graph $G^*$ of such a graph $G$ is a graph which has a vertex $p_j \in V(G^*)$ for each face (2-cell) $P_j$ of the CW 2-complex, and an edge joining $p_i$ and $p_j$ for each edge of $G$ in $\overline{P_i} \cap \overline{P_j}$ (if there are $k$ edges in $\overline{P_i} \cap \overline{P_j}$, then $[p_i, p_j]$ is a multiple edge of order $k$). By definition, every edge of $G^*$ has length 1.

Note that a CW 2-complex is a very general structure: if an edge $e$ belongs to the closure of $m_e$ faces, then $m_e$ can be any non-negative integer number; also, two edges in the boundary of a face can be “identified” in the CW complex.

Next, we deal with the main result of this section.
Theorem 4.4. Let $G$ be the 1-skeleton of a connected CW 2-complex $C$ and $G^*$ be its dual graph. Assume that every edge $e \in E(G)$ is included in the closure of a face of $C$ and satisfies $k_1 \leq L(e) \leq k_2$, every vertex $v \in V(G)$ satisfies $\deg(v) \leq \Delta$, every face of $G$ has at most $M$ edges and $G^*$ is a connected graph. Then $G$ is $\delta$-hyperbolic if and only if $G^*$ is $\delta'$-hyperbolic, quantitatively.

Proof. If $\Delta = 2$, then $C$ is just a face, $G$ is a cycle with $\delta(G) \leq k_2$ $M/4$ and $G^*$ is a vertex with $\delta(G^*) = 0$; hence, the result is trivial. Therefore without loss of generality we can assume that $\Delta \geq 3$. Let $G_0$ be a graph isomorphic to $G$ such that every edge of $G_0$ has length 1. Note that any isomorphism $g: G \rightarrow G_0$ is a bijective $(\max\{k_2, k^{-1}\}, 0)$-quasi-isometry; therefore, by Theorem 2.1, without loss of generality we can assume that every edge of $G$ has length 1.

First of all, assume that $G^*$ is a connected graph. We prove the result by proving that there exists a $(3/2)$-full $(\max\{2\Delta - 2, M/2\}, \max\{4\Delta - 4, M\})$-quasi-isometry $f: G \rightarrow G^*$.

Since the graph $G$ is the 1-skeleton of a CW 2-complex $C$ with faces $\{P_n\}$, then $\{p_n\} = V(G^*)$. If $e$ belongs to $\overline{P_i}$ for just one $i$, then we define $W(e) := p_i$; otherwise, we define $W(e)$ as the set of midpoints of the edges in $G^*$ corresponding to $e \in E(G)$, i.e., $W(e) := \{w(i,j) \in [p_ip_j]/e \in \overline{P_i} \cap \overline{P_j}, d_{G^*}(p_i, w(i,j)) = 1/2 = d_{G^*}(w(i,j), p_j)\}$. Note that, if there are $k \geq 2$ faces containing the edge $e$ in their closures, then $|W(e)| = k(k-1)/2$. For each $n$, we write $\overline{P_n} = \{e_1(n), \ldots, e_k(n)\}$ (note that $j_n \leq M$ by hypothesis). We denote by $W_n(e_i)$ the set of midpoints of the edges in $G^*$ starting in $p_n$ and corresponding to $e_i$, and by $T_n$ the set of indices with $W_n(e_i) \cap W_n(e_j) = \emptyset$ if and only if $e_i$ belongs to the closure of just one face. We define now $P_n^* := \cup_{j=1}^{j_n} \cup_{i \in T_n} [p_n, w(i,j)]$. It is clear that $G^* = \cup_n P_n^*$.

We define a function $f: G \rightarrow G^*$ as follows: if $e$ belongs to $\overline{P_i}$ for just one $i$, then we define $f(e) = p_i$; otherwise, we choose two faces $P_i, P_j$ with $e \in \overline{P_i} \cap \overline{P_j}$ and we define $f(e) = w(i,j)$; for each vertex $v \in V(G)$, let us choose an edge $e \in E(G)$ starting in $v$, and define $f(v)$ as the image via $f$ of the interior of $e$.

Let us consider $x, y \in \partial P_n$. If $f(x), f(y) \in P_n^*$, then $d_{G^*}(f(x), f(y)) \leq \text{diam}_{G^*}(P_n^*) = 1$. If we have $x \in P_n^*$ and $y \notin P_n^*$, then $d_{G^*}(f(x), f(y)) \leq \delta_{G^*}(P_n^*) = 1$. Now, if $x \notin P_n^*$ and $y \notin P_n^*$, then $d_{G^*}(f(x), f(y)) \leq 1$.

(4.3) $d_{G^*}(f(x), f(y)) \leq 2\Delta - 2$, for every $n$ and for all $x, y \in \partial P_n$.

Fix now $x, y \in G$ and a geodesic $\gamma$ in $G$ joining $x$ with $y$ (then, $L_G(\gamma) = d_G(x, y)$). Let $P$ be the set of collections of faces $P = \{P_{i_1}, P_{i_2}, \ldots, P_{i_r}\}$ with $\gamma \subset \cup_{m=1}^r \partial P_{i_m}$ and $\gamma \cap \partial P_{i_m}$ connected for every $m$; we say that $r$ is the size of the collection $P$ and we denote by $s(P) = r$. Let us consider $P' \in \mathbb{P}$ with $s(P') = \min_{P \in \mathbb{P}} s(P) = k$. Denote by $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ the faces in $P'$; without loss of generality we can assume that $\gamma$ meets $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$ in this order (with $x \in \partial P_{i_1}, y \in \partial P_{i_k}$); note that it is possible to have $i_a = i_b$ with $a \neq b$. Define $\gamma_j$ as the connected subgeodesic of $\gamma$ such that $\gamma_j \subseteq \gamma \cap \partial P_{i_j}$ $(1 \leq j \leq k)$, $\gamma_i \cap \gamma_{i+1} \neq \emptyset$ $(1 \leq j \leq k-1)$ if $k > 1$, and $\gamma = \gamma_1 \cup \cdots \cup \gamma_k$. Note that $L_G(\gamma_j) \geq 1$ for $1 < j < k$, $L_G(\gamma_1) > 0$ and $L_G(\gamma_k) > 0$.

If $k = 1$, then (3.3) gives $d_{G^*}(f(x), f(y)) \leq 2\Delta - 2$.

If $k = 2$ and $z \in \gamma_1 \cap \gamma_2$, then we have

$$d_{G^*}(f(x), f(y)) \leq d_{G^*}(f(x), f(z)) + d_{G^*}(f(z), f(y)) \leq 2\Delta - 2 + 2\Delta - 2 = 4\Delta - 4.$$
If \( k \geq 3 \) and \( z_j \in \gamma_j \cap \gamma_{j+1} \) for \( 1 \leq j \leq k - 2 \) then we have \( \text{d}_G(z_j, z_{j+1}) \geq 1 \) and

\[
\text{d}_G(f(x), f(y)) \leq \text{d}_G(f(x), f(z_1)) + \sum_{j=1}^{k-2} \text{d}_G(f(z_j), f(z_{j+1})) + \text{d}_G(f(z_{k-1}), f(y))
\]

\[
\leq 2 \Delta - 2 + \sum_{j=1}^{k-2} (2 \Delta - 2) + 2 \Delta - 2 \leq 4 \Delta - 4 + (2 \Delta - 2) \sum_{j=1}^{k-2} \text{d}_G(z_j, z_{j+1})
\]

\[
= 4 \Delta - 4 + (2 \Delta - 2) \text{d}_G(z_1, z_{k-1}) \leq 4 \Delta - 4 + (2 \Delta - 2) \text{d}_G(x, y).
\]

Let us consider a geodesic \( \gamma^* \) in \( G^* \) joining \( f(x) \) with \( f(y) \) (then, \( L_{\gamma^*}(\gamma^*) = \text{d}_G(f(x), f(y)) \)). Note that \( f(x) \) (respectively, \( f(y) \)) is either a midpoint of one edge in \( E(G^*) \) or a vertex in \( V(G^*) \). If \( f(x) = f(y) \) then there exists \( i \) such that \( x, y \in \partial P_i \); since every face of \( G \) has at most \( M \) edges, we have \( \text{d}_G(x, y) \leq M/2 \).

Then, \( \text{d}_G(f(x), f(y)) = 0 \geq \text{d}_G(x, y) - M/2 \). We assume now that there exist \( p_1, p_2, \ldots, p_m \in V(G^*) \) such that \( \gamma^* \) meets \( f(x), p_1, \ldots, p_m, f(y) \) in this order (with \( 0 \leq \text{d}_G(f(x), p_1), \text{d}_G(p_m, f(y)) \leq 1/2 \)) and we have \( \text{d}_G(f(x), f(y)) = m \) (if \( f(x), f(y) \) are midpoints of edges), \( \text{d}_G(f(x), f(y)) = m + 1/2 \) (if just one is a midpoint of edge) or \( \text{d}_G(f(x), f(y)) = m - 1 \) (if \( f(x), f(y) \) in \( V(G^*) \)).

If \( m = 1 \), then we have that \( f(x), f(y) \in P^*_i \) and \( x, y \in \partial P_i \); so, we have \( \text{d}_G(x, y) \leq M/2 \). Therefore, \( \text{d}_G(f(x), f(y)) \geq 0 \geq \text{d}_G(x, y) - M/2 \).

Assume now that \( m \geq 2 \). Let \( w_n := w^{(1, n+1)} \in \gamma^* \) be the midpoint of the edge \( [p_n, p_{n+1}] \), for \( 1 \leq n \leq m - 1 \). Let us consider an edge \( e_n \subseteq P_n \cap P_{n+1} \), for \( 1 \leq n \leq m - 1 \); let \( z_n \) be the midpoint of \( e_n \), for \( 1 \leq n \leq m - 1 \). Then, for \( 1 \leq n \leq m - 1 \), we have \( \text{d}_G(z_n, z_{n+1}) \leq M/2 \), \( \text{d}_G(w_n, w_{n+1}) = 1 \), and

\[
\text{d}_G(f(x), f(y)) = \text{d}_G(f(x), w_1) + \sum_{n=1}^{m-2} \text{d}_G(w_n, w_{n+1}) + \text{d}_G(w_{m-1}, f(y))
\]

\[
= \text{d}_G(f(x), w_1) + \sum_{n=1}^{m-2} 1 + \text{d}_G(w_{m-1}, f(y))
\]

\[
\geq \text{d}_G(x, z_1) - M/2 + 2 \sum_{n=1}^{m-2} M/2 + \text{d}_G(z_{m-1}, y) - M/2
\]

\[
\geq \frac{2}{M} \text{d}_G(x, z_1) + \frac{2}{M} \sum_{n=1}^{m-2} \text{d}_G(z_n, z_{n+1}) + \frac{2}{M} \text{d}_G(z_{m-1}, y) - M
\]

\[
\geq \frac{2}{M} \text{d}_G(x, y) - M.
\]

Consequently, \( f \) is a \((\max\{2 \Delta - 2, M/2\}, \max\{4 \Delta - 4, M\})\)-quasi-isometric embedding. Furthermore, \( f \) is \((3/2)\)-full: since for every \( e \in E(G) \) we have

\[
\text{diam}_{G^*}(W(e)) \leq 1, \quad W(e) \cap f(G) \neq \emptyset \quad \text{and} \quad \sup_{x \in G^*} \text{d}_G(x, \cup_{e \in E(G^*)} W(e)) = 1/2.
\]

This finishes the proof by Theorem 2.1. \( \square \)

The following examples show that the conclusion of Theorem 4.4 does not hold if we remove any hypothesis from its statement. In order to compute the hyperbolicity constant of these examples, we need an additional result.

Given a graph \( G \), we say that a family of subgraphs \( \{G_n\}_n \) of \( G \) is a \textit{tree-decomposition} of \( G \) if \( \cup_n G_n = G \), \( G_n \cap G_m \) is either a vertex or the empty set for each \( n \neq m \), and if the graph \( R \) defined as follows is a tree: for each \( n \) let us define a point \( v_n \) (\( v_n \) is an abstract point, it is not contained in \( G \)); we have \( V(R) := \{v_n\}_n \) and \([v_n, v_m] \in E(R)\) if and only if \( G_n \cap G_m \neq \emptyset \).
The following result allows to obtain global information about the hyperbolicity of a graph from local information (see [3, Theorem 5]).

**Theorem 4.5.** Let $G$ be a graph and $\{G_n\}$ a tree-decomposition of $G$. Then $\delta(G) = \sup_n \delta(G_n)$.

**Example 4.6.** Let us consider the sequence of wheel graphs $\{W_n\}_{n=4}^{\infty}$ ($W_n$ has $n$ vertices). Choose two vertices $a_n^0, b_n^0 \in V(W_n)$ (different from the central vertex of $W_n$) with $d_{W_n}(a_n^0, b_n^0) = 1$ for $n \geq 5$, and two vertices $a_{n+1}^0, b_{n+1}^0 \in V(W_n)$ (different from the central vertex of $W_n$) with $d_{W_n}(a_{n+1}^0, b_{n+1}^0) = 1$ for $n \geq 4$ and $\{a_{n+1}^0, b_{n+1}^0\} \cap \{a_n^0, b_n^0\} = \emptyset$ for $n \geq 5$. We define $G$ as the union of $\{W_n\}_{n=4}^{\infty}$ obtained by identifying $[a_{n+1}^0, b_{n+1}^0]$ with $[a_n^0, b_n^0]$ for $n \geq 4$. Since the central vertex of each $W_n$ has degree $n - 1$, the degree of $G$ is not bounded. It is clear that $G$ is quasi-isometric to the graph $G'$ obtained as the union of $\{W_n\}_{n=4}^{\infty}$ by identifying $a_{n+1}^0$ with $a_n^0$ for $n \geq 4$. Theorems 4.5 and 2.3 give that $G'$ is hyperbolic, since $\delta(G') = \sup_n \delta(W_n) = 3/2$. Hence, $G$ is also hyperbolic by Theorem 2.1.

Its dual graph $G^*$ is isometric to a union of cycle graphs $\{C_n\}_{n=3}^{\infty}$ such that each $C_n$ is joined with $C_{n+1}$ by a graph isometric to the path graph $P_2$ for $n \geq 3$. Theorems 4.5 and 2.3 give that $G^*$ is not hyperbolic, since $\delta(G^*) = \sup_n \delta(C_n) = \sup_n n/4 = \infty$, although $G$ is hyperbolic.

**Example 4.7.** Let us consider the sequence of graphs $\{C_n \times P_2\}_{n=3}^{\infty}$ represented in $\mathbb{R}^2$ by an “exterior” copy of $C_n$ joined with an “interior” copy of $C_n$ by $n$ edges. Choose two vertices $a_n^0, b_n^0 \in V(C_n \times P_2)$ (the exterior copy of $C_n$) with $d_{C_n \times P_2}(a_n^0, b_n^0) = 1$ for $n \geq 4$, and two vertices $a_{n+1}^0, b_{n+1}^0 \in V(C_n \times P_2)$ (the interior copy of $C_n$) with $d_{C_n \times P_2}(a_{n+1}^0, b_{n+1}^0) = 1$ for $n \geq 3$ and $\{a_{n+1}^0, b_{n+1}^0\} \cap \{a_n^0, b_n^0\} = \emptyset$ for $n \geq 4$. We define $G$ as the union of $\{C_n \times P_2\}_{n=3}^{\infty}$ obtained by identifying $[a_{n+1}^0, b_{n+1}^0]$ with $[a_n^0, b_n^0]$ for $n \geq 3$. Note that the “central face” of each $C_n \times P_2$ (whose boundary is the interior copy of $C_n$) has $n$ edges, and therefore there is not an upper bound for the number of edges of the faces in $G$. It is clear that $G$ is quasi-isometric to the graph $G'$ which is the union of $\{C_n \times P_2\}_{n=3}^{\infty}$ obtained by identifying $a_{n+1}^0$ with $a_n^0$ for $n \geq 3$. Since $C_n \times P_2$ has an isometric subgraph which is isomorphic to $C_n$, Lemma 2.4 gives that $\delta(C_n \times P_2) \geq \delta(C_n)$. Theorems 4.5 and 2.3 give that $G'$ is not hyperbolic, since $\delta(G') = \sup_n \delta(C_n \times P_2) \geq \sup_n \delta(C_n) = \sup_n n/4 = \infty$. Hence, $G$ is not hyperbolic.

Its dual graph $G^*$ is isometric to a union of wheel graphs $\{W_n\}_{n=4}^{\infty}$ such that each $W_n$ is joined with $W_{n+1}$ by a graph isometric to the path graph $P_2$ for $n \geq 3$. Theorems 4.5 and 2.3 give that $G^*$ is hyperbolic, since $\delta(G^*) = \sup_n \delta(W_n) = 3/2$, although $G$ is not hyperbolic.

**Example 4.8.** Let us consider the CW 2-complex with only one 2-cell, the open unit square $\{(x, y) \in \mathbb{R}^2 | 0 < x < 1, 0 < y < 1\}$, and with 1-skeleton equal to the Cayley graph of $\mathbb{Z}^2$, i.e., the planar graph $G$ with $V(G) := \mathbb{Z}^2$ and unit edges defined by $E(G) := \{((a, b), (c, d)) | |a - c| + |b - d| = 1\}$ (each edge is represented by a straight line). Let $G^*$ be the dual graph of $G$. Since $G$ is quasi-isometric to $\mathbb{R}^2$, $G$ is not hyperbolic by Theorem 2.1. Note that just 4 edges of $G$ belong to the closure of the single face. However, $G^*$ is connected and 0-hyperbolic, since it is a graph with just one vertex.

**Example 4.9.** Let us consider the Hausdorff space $X \subset \mathbb{R}^2$ defined by $\{(x, y) \in \mathbb{R}^2 | p \leq x \leq p + 1, q \leq y \leq q + 1, \text{ for every } p, q \in \mathbb{Z} \text{ such that } p + q \text{ is even}\}$ (X looks like an infinite chessboard). Now, consider the natural CW 2-complex associated to $X$ and let $G$ be its 1-skeleton. Note that $G^*$ is not connected and each connected component of $G^*$ is a single vertex (and then 0-hyperbolic) since $E(G^*) = \emptyset$; however, $G$ is not hyperbolic, since it is equal to the graph in the previous example.

Theorems 4.4 and 4.5 allows to deduce the following generalization of Theorem 4.4.

**Theorem 4.10.** Let $G$ be the 1-skeleton of a connected CW 2-complex $C$ and $G^*$ be its dual graph. Assume that every edge $e \in E(G)$ is included in the closure of a face of $C$ and satisfies $k_1 \leq L(e) \leq k_2$, every vertex $v \in V(G)$ satisfies $\deg(v) \leq \Delta$, every face of $C$ has at most $M$ edges and there exists a tree-decomposition $\{G_n\}$ of $G$ such that $\{G_n^*\}$ are the connected components of $G^*$. Then $G$ is $\delta^*$-hyperbolic if and only if $G_n^*$ is $\delta^*$-hyperbolic for every $n$, quantitatively.
Proof. Since \( \{G_n\} \) is a tree-decomposition of \( G \), we have by Theorem 4.5 that \( \delta(G) = \sup_n \delta(G_n) \).

Assume first that \( G \) is \( \delta \)-hyperbolic. Then \( G_n \) is \( \delta \)-hyperbolic for every \( n \). Since \( G_n^* \) is connected for every \( n \), Theorem 4.4 gives that \( G_n^* \) is \( \delta^* \)-hyperbolic for every \( n \), where \( \delta^* \) depends just on \( k_1, k_2, \Delta, M \) and \( \delta \).

Assume now that \( G_n^* \) is \( \delta^* \)-hyperbolic for every \( n \). Then Theorem 4.4 gives that \( G_n \) is \( \delta \)-hyperbolic for every \( n \), where \( \delta \) depends just on \( k_1, k_2, \Delta, M \) and \( \delta^* \). Then \( G \) is \( \delta \)-hyperbolic.

\[ \square \]

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Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain  
E-mail address: wcarball@math.uc3m.es  

St. Louis University (Madrid Campus), Avenida del Valle 34, 28003 Madrid, Spain  
E-mail address: aportil2@slu.edu  

Departamento de Matemáticas, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain  
E-mail address: jomaro@math.uc3m.es  

Facultad de Matemáticas, Universidad Autónoma de Guerrero, Carlos E. Adame No.54 Col. Garita, 39650 Acalpulco GRO., Mexico  
E-mail address: josemariasigarretalnira@yahoo.es