QUASI-ISOMETRIES AND ISOPERIMETRIC INEQUALITIES IN PLANAR DOMAINS

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Abstract. This paper studies the stability of isoperimetric inequalities under quasi-isometries between non-exceptional Riemann surfaces endowed with their Poincaré metrics. This stability was proved by Kanai in a more general setting under the condition of positive injectivity radius. The present work proves the stability of the linear isoperimetric inequality for planar surfaces (genus zero surfaces) without any condition on their injectivity radii. It is also shown that any non-linear isoperimetric inequality implies positive injectivity radius for the surface and therefore the stability of any isoperimetric inequality.

Key words and phrases: Riemann surface; Poincaré metric; isoperimetric inequality; linear isoperimetric inequality; quasi-isometry.

1. Introduction

An interesting problem in the study of geometric properties of surfaces is to consider their stability under appropriate deformations. In the 1985, in [18] M. Kanai proved the quasi-isometric stability (see the definition of quasi-isometry in Section 2) of several geometric properties for a large class of Riemannian manifolds.

We shall be interested not only in his results but in the ideas behind the proofs. Concretely, those relating the manifold with a particular graph (an $\varepsilon$-net of the manifold) in order to study the stability of the quasi-isometry. Several authors have followed Kanai in studying the stability of some other property, or in proving the equivalence of a manifold with a different associated graph (see, e.g., [1], [7], [14], [17], [19], [20], [26], [27], [30], [31], [33], [35], [37]).

Quasi-isometries play a central role in the theory of Gromov hyperbolic spaces for they preserve hyperbolicity of geodesic metric spaces (see, e.g., [15], [16]).

A non-exceptional Riemann surface $S$ will mean a two-dimensional manifold with a complete conformal metric of constant negative curvature $-1$. In this case, the universal covering space of $S$ is the unit disk $\mathbb{D}$ endowed with its Poincaré metric. The only exceptional Riemann surfaces are the sphere, the plane, the punctured plane and the tori.

A Riemann surface $S$ satisfies the $\alpha$-isoperimetric inequality ($1/2 \leq \alpha \leq 1$) if there exists a constant $c_\alpha(S)$ such that

$$A_S(\Omega)^{\alpha} \leq c_\alpha(S)L_S(\partial \Omega)$$

for every relatively compact domain $\Omega \subset S$. Throughout, $A_S$, $L_S$ and $d_S$ refer to Poincaré area, length and distance of $S$ and LII refers to the 1-isoperimetric inequality also known as the linear isoperimetric inequality.

There are close connections between LII and some conformal invariants of Riemann surfaces, namely the bottom of the spectrum of the Laplace-Beltrami operator, the exponent of convergence, and the Hausdorff...
dimensions of the sets of both bounded geodesics and escaping geodesics in the surface (see [5], [6, p.228], [10], [11], [12], [13], [22], [23], [36, p.333]). Isoperimetric inequalities are of interest in pure and applied mathematics (see, e.g., [9], [25]).

The injectivity radius \( \iota(p) \) of \( p \in S \) is defined as the supremum of those \( r > 0 \) such that \( B_S(p,r) \) is simply connected or, equivalently, as half the infimum of the lengths of the (homotopically non-trivial) loops based at \( p \). The injectivity radius \( \iota(S) \) of \( S \) is the infimum over \( p \in S \) of \( \iota(p) \).

This paper considers the stability of isoperimetric inequalities under quasi-isometries between non-exceptional Riemann surfaces. This stability was proved by Kanai in [18] under the hypothesis \( \iota(S) > 0 \) in a very general setting. Example 2.1 in the next section shows that the stability fails without the hypothesis \( \iota(S) > 0 \). Since this example involves non-zero genus surfaces, it is natural to wonder if the stability holds for planar surfaces.

The main result in this paper is the following.

**Theorem 1.1.** Let \( S \) and \( S' \) be quasi-isometric non-exceptional genus zero Riemann surfaces. Then \( S' \) satisfies the linear isoperimetric inequality if and only if \( S \) satisfies the linear isoperimetric inequality. Furthermore, if \( f : S \rightarrow S' \) is a \( c \)-full \((a,b)\)-quasi-isometry, and \( c_1(S') < \infty \) then \( c_1(S) \leq C \), where \( C \) is a universal constant which just depends on \( a, b, c \) and \( c_1(S') \).

For surfaces of positive finite genus, Theorem 7.2 shows that the first conclusion of Theorem 1.1 holds; however Example 7.3 shows that the second conclusion of Theorem 1.1 fails in this case.

The idea behind the proof of Theorem 1.1 is simple: each surface is split into a thin part (with small injectivity radius) and a thick part; a slight modification of the proof of Kanai’s Theorem applied to the thick part, together with some new arguments to show that the thin part is “essentially” preserved under the quasi-isometry give the theorem. The difficulty is the following: two quasi-isometric surfaces have a similar shape at a large scale (if viewed from sufficiently far), but they can look very different at a small scale (by definition a quasi-isometry may not be continuous). In particular, the image of a continuous loop by a quasi-isometry need not be a continuous curve, and thus the injectivity radii can be very different in two quasi-isometric surfaces (see, e.g., Examples 2.2 and 2.3). Theorem 5.1 deals with this situation and states that a quasi-isometry between planar surfaces maps points with small injectivity radius to points with small injectivity radius (in a precise quantitative way). In fact, the core of this work is devoted to proving Theorem 5.1.

A very different situation appears when dealing with the \( \alpha \)-isoperimetric inequality, \( 1/2 \leq \alpha < 1 \). Theorem 8.2 states that, in this case, if \( S \) and \( S' \) are quasi-isometric with \( \iota(S) > 0 \), then \( S' \) satisfies the \( \alpha \)-isoperimetric inequality if and only \( S \) satisfies the \( \alpha \)-isoperimetric inequality and \( \iota(S') > 0 \). Note that here we have no hypothesis on genus.

Hence, the behavior of the \( \alpha \)-isoperimetric inequality in Riemann surfaces under quasi-isometries is very different in the cases \( \alpha = 1 \) and \( \alpha < 1 \).

The outline of this paper is as follows. Section 2 contains some background and examples. In Section 3 the continuity of the injectivity radius in a Riemann surface is studied. Section 4 contains some technical lemmas on quasi-isometries which will be needed in Section 5 in order to control the distortion of the injectivity radius under quasi-isometries. In Section 6 the proof of Theorem 1.1 is given, and finally, sections 7 and 8 are devoted to generalize this theorem to finite genus surfaces and to non-linear isoperimetric inequalities, respectively.

## 2. Background and examples

A function between two metric spaces \( f : X \rightarrow Y \) is said to be an \((a,b)\)-quasi-isometric embedding with constants \( a \geq 1, b \geq 0 \), if

\[
\frac{1}{a} d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a d_X(x_1, x_2) + b, \quad \text{for every } x_1, x_2 \in X.
\]

Such a quasi-isometric embedding \( f \) is a quasi-isometry if, furthermore, there exists a constant \( c \geq 0 \) such that \( f \) is \( c \)-full, i.e., if for every \( y \in Y \) there exists \( x \in X \) with \( d_Y(y, f(x)) \leq c \).
Two metric spaces $X$ and $Y$ are quasi-isometric if there exists a quasi-isometry between them.

An $(a, b)$-quasigeodesic in $X$ is an $(a, b)$- quasi-isometric embedding between an interval of $\mathbb{R}$ and $X$. A geodesic in $X$ is a $(1, 0)$-quasigeodesic.

It is easy to check that to be quasi-isometric is an equivalence relation on the set of metric spaces.

The word geodesic will always be used with this meaning except for the case of either simple closed geodesics (which are just local geodesics) or geodesic loops (which are just local geodesics except in their basepoints).

The surface $S$ will be split into thin and thick parts, and some standard tools for constructing Riemann surfaces will be needed. Doubly connected domains will be crucial.

A collar in a non-exceptional Riemann surface $S$ about a simple closed geodesic $\sigma$ is a doubly connected domain in $S$ “bounded” by two Jordan curves (called the boundary curves of the collar) orthogonal to the pencil of geodesics emanating from $\sigma$; such collar is equal to \{ $p \in S : d_S(p, \sigma) \leq d$\}, for some positive constant $d$. The constant $d$ is called the width of the collar.

Let $S$ be a non-exceptional Riemann surface with a cusp $q$ (if $S \subset \mathbb{C}$, every isolated point in $\partial S$ is a cusp). A collar in $S$ about $q$ is a doubly connected domain in $S$ “bounded” both by $q$ and a Jordan curve (called the boundary curve of the collar) orthogonal to the pencil of geodesics emanating from $q$. It is well known that the length of the boundary curve is equal to the area of the collar (see, e.g., [4]). A collar of area $\beta$ is called a $\beta$-collar.

A $Y$-piece is a compact bordered Riemann surface which is topologically a sphere without three disks and whose border is the union of three simple closed geodesics. Given three positive numbers $a$, $b$, $c$, there is a unique (up to conformal mapping) $Y$-piece such that their boundary curves have lengths $a$, $b$, $c$ (see, e.g., [29, p.410]). They are a standard tool for constructing Riemann surfaces ([8, Chapter X.3] and [6, Chapter 1]).

A generalized or degenerated $Y$-piece is a bordered or non-bordered Riemann surface which is topologically a sphere without open disks and $m$ points, with integers $n, m \geq 0$ and $n + m = 3$, so that the $n$ boundary curves are simple closed geodesics and the $m$ deleted points are cusps. Observe that a generalized $Y$-piece is topologically the union of a $Y$-piece and $m$ cylinders, with $0 \leq m \leq 3$.

A funnel is a bordered Riemann surface which is topologically a cylinder and whose border is a simple closed geodesic. Given any positive number $a$, there is a unique (up to conformal mapping) funnel such that its boundary curve has length $a$.

The following example shows that the stability of LII fails for surfaces with zero injectivity radius.

Example 2.1. There exist two non-exceptional Riemann surfaces $S, S'$ and a c-full $(a, b)$-quasi-isometry $f : S \to S'$, such that $\iota(S) = \iota(S') = 0$, $S$ does not satisfy the LII and $S'$ satisfies the LII.

Let us consider two isometric $Y$-pieces $Y_1, Y_2$ such that $\partial Y_j$ is the union of three simple closed geodesics with length $1$ for $j = 1, 2$. Denote by $W$ the bordered surface obtained by pasting two boundary curves of $Y_1$ with two boundary curves of $Y_2$ (X is a torus with two holes). Let us consider a sequence $\{X_m\}_{m \geq 1}$ of bordered surfaces isometric to $X$; denote by $S_0$ the bordered surface obtained by pasting a boundary curve of $X_m$ with a boundary curve of $X_{m+1}$ for every $m \geq 1$. Consider now a generalized $Y$-piece $Y_0$ with a cusp and such that $\partial Y_0$ is the union of two simple closed geodesics with length $1$.

$S$ is the (non bordered) surface obtained by pasting a funnel (with boundary of length 1) to one boundary curve of $Y_0$ and $S_0$ to the other boundary curve of $Y_0$. $S$ does not satisfy the LII since $\cup_{m=1}^{n} X_m$ has area $4\pi n$ and its boundary has length 2 for every $n \geq 1$.

The surface $S'$ is obtained by pasting a funnel (with boundary of length 1) to a generalized $Y$-piece $Y^*$ with two cusps and such that $\partial Y^*$ is a simple closed geodesic with length 1. $S'$ satisfies the LII since a surface of finite type satisfies the LII if and only if it has at least a funnel.

The following examples show that the conclusion of Theorem 5.1 does not hold if $S$ or $S'$ are surfaces of positive genus. In particular, Example 2.2 shows that thin parts are not in correspondence.
Example 2.2. There exist constants $a, b, c, I_1, I_2$ with the following property: for each $n$ there exist non-exceptional Riemann surfaces $S_n, S_n'$ and a $c$-full $(a, b)$-quasi-isometry $f_n : S_n \to S_n'$, such that $\iota(z) \leq n$ for every $z \in S_n$ and $I_1 \leq \iota(z) \leq I_2$ for every $z \in S_n'$.

Let $S_n$ be the annulus with the simple closed geodesic with length $2n$. Consider an 1-net $N_n$ of $S_n$; by [18] there exists a quasi-isometry $g_n : S_n \to N_n$ with universal constants. By [24, Theorem 3.4] (see also [3, Theorem 2]), there exist a cubic graph $C_n$ and a quasi-isometry $h_n : N_n \to C_n$ with universal constants. Let us consider a sequence $\{Y_m\}$ of $Y$-pieces such that $\partial Y_m$ is the union of three simple closed geodesics with length 1 for every $m$ (therefore, they are isometric). It suffices to consider as $S_n'$ the surface obtained by pasting the $Y$-pieces $\{Y_m\}$ following the combinatorial design of $C_n$.

Example 2.3. There exist constants $a, b, c, I_1, I_2$ with the following property: there exist non-exceptional Riemann surfaces $S, S'$ and a $c$-full $(a, b)$-quasi-isometry $f : S \to S'$, such that $I_1 \leq \iota(z) \leq I_2$ for every $z \in S$ and $\iota(S') = 0$.

Let us consider two isometric $Y$-pieces $Y_1, Y_2$ such that $\partial Y_j$ is the union of three simple closed geodesics with length 1 for $j = 1, 2$. Denote by $X$ the bordered surface obtained by pasting two boundary curves of $Y_1$ with two boundary curves of $Y_2$ ($X$ is a torus with two holes). Let us consider a sequence $\{X_m\}_{m \in \mathbb{Z}}$ of bordered surfaces isometric to $X$; then $S$ is obtained by pasting a boundary curve of $X_n$ with a boundary curve of $X_{n+1}$ for every $n \in \mathbb{Z}$.

Consider now two isometric generalized $Y$-pieces $Y_3, Y_4$ with a cusp and such that $\partial Y_j$ is the union of two simple closed geodesics with length 1 for $j = 3, 4$. It suffices to consider $S'$ as the (non bordered) surface obtained by pasting the two boundary curves of $Y_3$ with the two boundary curves of $Y_4$ ($S'$ is a torus with two cusps).

3. Continuity of the injectivity radius

The following result is well-known and easy to check.

**Lemma 3.1.** Let $M$ be a Riemannian manifold and $x, y \in M$. Then $|\iota(x) - \iota(y)| \leq d_M(x, y)$.

This last result can be improved for small values of the injectivity radius.

**Lemma 3.2.** Let $S$ be a non-exceptional Riemann surface and $z, w \in S$. Then

$$\iota(w) \geq \arcsinh \left( e^{-d_S(z, w)} \min \{1, \sinh(\iota(z))\} \right).$$

In particular, if $\sinh(\iota(z)), \sinh(\iota(w)) \leq 1$, then $|\log \sinh(\iota(w)) - \log \sinh(\iota(z))| \leq d_S(z, w)$.

**Proof.** Let us choose geodesic loops $\gamma_z$ and $\gamma_w$ with respective base points $z$ and $w$ such that $\iota(z) = L_S(\gamma_z)$ and $\iota(w) = L_S(\gamma_w)$.

Assume first that $\gamma_z$ and $\gamma_w$ are freely homotopic. It is clear that the minimum value of $\iota(w)$ is attained if $\gamma_z$ and $\gamma_w$ bordered a cusp and $z$ and $w$ belong to the same geodesic escaping to the cusp.

As usual, consider a fundamental domain for $S$ in the upper halfplane $\mathbb{H}$ contained in $\{ z \in \mathbb{H} : 0 \leq \Re z \leq 1 \}$ and such that $\{ z \in \mathbb{H} : 0 \leq \Re z \leq 1, \Im z \geq 1/2 \}$ corresponds to the 2-collar of this cusp in $S$. Let us represent $\gamma_z$ (respectively, $\gamma_w$) in the upper half-plane by means of a geodesic with endpoints $i\alpha$ and $i\alpha + 1$ (respectively, $i\beta$ and $i\beta + 1$, with $\beta > \alpha$). Note that

$$\sinh(\iota(z)) = \sinh \left( \frac{d_\mathbb{H}(i\alpha, i\alpha + 1)}{2} \right) = \frac{1}{2\alpha}, \quad \sinh(\iota(w)) = \frac{1}{2\beta}, \quad d_\mathbb{H}(z, w) = d_\mathbb{H}(i\alpha, i\beta) = \log \frac{\beta}{\alpha} = \log \frac{\sinh(\iota(z))}{\sinh(\iota(w))}.$$

Hence, the minimum value of $\iota(w)$ is attained with $\iota(w) = \arcsinh \left( e^{-d_\mathbb{H}(z, w)} \sinh(\iota(z)) \right)$.

Assume now that $\gamma_z$ and $\gamma_w$ are not freely homotopic. Let us consider a geodesic $[z, w]$ in $S$ and the nearest point $z_0$ to $z$ in $[z, w]$ with a geodesic loop $\gamma_{z_0}$ freely homotopic to $\gamma_w$ such that $\iota(z_0) = L_S(g_{z_0})$. It is not difficult to see that $\iota(z_0) \geq \arcsinh 1$ (the injectivity radius of any point in the boundary of the 2-collar of a cusp). The previous argument gives $\iota(w) \geq \arcsinh \left( e^{-d_\mathbb{H}(z_0, w)} \sinh(\iota(z_0)) \right) \geq \arcsinh e^{-d_\mathbb{H}(z, w)}$. This finishes the proof of the first statement. The second statement is a direct consequence of the first one. \( \square \)
4. Technical lemmas on quasi-isometries

A key step in the proof of our main result in this paper (Theorem 1.1) is to control the distortion of the injectivity radius under quasi-isometric transformations (see Theorems 5.1 and 5.2). Due to the complexity of the proofs of these Theorems, this section is devoted to present some technical lemmas used in their proofs.

Along this chapter, $\sigma$ will denote a simple closed geodesic in $S$ and $w$ the width of the collar of $\sigma$, where $\cosh w = \coth(L_S(\sigma)/2)$.

Let us consider $H > 0$, a metric space $X$, and a subset $Y \subseteq X$. The set $V_H(Y) := \{x \in X : d(x, Y) \leq H\}$ is called the $H$-neighborhood of $Y$ in $X$.

A control on how collars behave under quasi-isometries will be needed, and thus a more general definition is required: Let us consider a finite or infinite geodesic $\gamma \subset S$ and a connected subset $\gamma_0$ of that geodesic. Given two positive constants $h$ and $r$, then the $h$-neighborhood of $f(\gamma)$ in $S'$ is an $(f, \gamma, \gamma_0, h, r)$-tube $T$ if for every point $p \in \gamma_0$, the closed ball $B_{S'}(f(p), r)$ is contained in $T$.

In principle, although a tube does not need to be doubly connected, Theorems 4.4 and 4.5 will show that they are "essentially" doubly connected, and that explain the name.

**Remark 4.1.** The Collar Lemma states that if $\sigma$ is a simple closed geodesic there exists a collar about $\sigma$ of width $d$, for every $0 < d \leq w$, where $\cosh w = \coth(L_S(\sigma)/2)$. Hence, if $L_S(\sigma) < 2\arccoth(\cosh t)$, then $w > t$.

Denote by $C_{\sigma, d}$ the collar of $\sigma$ of width $d$ and by $C_{\sigma}$ the collar of $\sigma$ of width $w$. It is well known that if $\sigma_1$ and $\sigma_2$ are disjoint simple closed geodesics, then $C_{\sigma_1} \cap C_{\sigma_2} = \emptyset$.

For each cusp there exists a 2-collar and 2-collars of different cusps are disjoint. Besides, the collar $C_{\sigma}$ of the simple closed geodesic $\sigma$ does not intersect the 2-collar of a cusp (see [28], [34] and [6, Chapter 4]). If a $\lambda$-collar of a cusp (with $0 < \lambda \leq 2$) in a Riemann surface has boundary curve $\alpha$, denote this collar by $C_{\alpha}$. Denote also by $C(\sigma)_{H}$ the $H$-neighborhood of the 2-collar of a cusp with boundary $\alpha$ (now $\alpha$ can be a union of closed curves).

The next result deals with collars of geodesics and cusps separately:

**Lemma 4.2.** Assume that $S$ is a genus zero Riemann surface.

1. Let $t > 0$, and $\gamma$ be any geodesic perpendicular to $\sigma$ contained in $S$ with $L_S(\gamma) = 2w$. Then, there exist positive constants $k_1, k_2$, which just depend on $a, b, c, t$, so that for $\gamma_0 := \{p \in \gamma : d_S(p, \sigma) < w - k_1\}$ and $L_S(\sigma) < k_2$ there exists an $(f, \gamma, \gamma_0, h, h + t)$-tube $T \subset S'$ with $h := 3a + b + c$.

2. Let $C_{\alpha}$ be the 2-collar of a cusp in $S$ with boundary curve $\sigma$, $\gamma$ an infinite geodesic contained in $C_{\alpha}$ perpendicular to $\sigma$ and $t > 0$. Then, there exists a positive constant $k$, which just depends on $a, b, c, t, s$ and such that for $\gamma_0 := \{p \in \gamma : d_S(p, \sigma) > k\}$ there exists an $(f, \gamma, \gamma_0, h, h + t)$-tube $T \subset S'$ with $h := 2a + b + c$.

**Proof.** (1) Set $k_1 := 2a(3a + 2b + 2c + t)$ and $k_2 := 2\arccoth(\cosh k_1)$. Notice that $k_1 \geq 6$, since $a \geq 1$. Since $S$ is a zero genus surface, $\gamma$ is a geodesic (not just a local geodesic); therefore, $f(\gamma)$ is an $(a, b)$-quasi-geodesic.

Seeking for a contradiction, let us assume that there exists a point $p \in \gamma_0$ such that the ball $B := B_{S'}(f(p), h + t)$ is not contained in $T$. That is, there exists a point $q \in B \setminus T$ for which

$$d_{S'}(f(\gamma)) > h.$$  

(4.2)

Since $f$ is $c$-full, there must exist $p_1 \in S \setminus \gamma$ such that $d_{S'}(f(p_1), q) \leq c$. Let us assume that $d_S(p_1, \sigma) > w - k_1/2$. Since $p \in \gamma_0$, it means that $d_S(p, p_1) > k_1/2$. Using the fact that $f$ is an $(a, b)$-quasi-isometry,

$$d_{S'}(f(p), f(p_1)) > \frac{1}{a}d_S(p, p_1) - b > \frac{k_1}{2a} - b.$$  

(4.3)
By the triangle inequality, and using that \( q \in B \),
\[
(4.4) \quad d_S(f(p_1), f(p_2)) \leq d_S(f(p_1), q) + d_S(q, f(p)) \leq 3a + b + 2c + t.
\]
Combining now (4.3) and (4.4), one deduces \( k_1 < 2a(3a + 2b + 2c + t) \), which contradicts the definition of \( k_1 \). Therefore, \( p_1 \in C_{\sigma, w - k_1/2} \). Then, there exists a point \( p_2 \in \gamma \) close enough to \( p_1 \), verifying that \( d_S(p_1, p_2) \) is upper bounded by the length of one of the boundary curves of \( C_{\sigma, w - k_1/2} \).

Using Fermi coordinates based on \( \sigma \), it is easy to check that \( L_S(\partial C_{\sigma, w - k_1/2})/2 \leq L_S(\partial C_{\sigma})/2 = L_S(\sigma) \cosh w \). Remark 4.1, gives \( d_S(p_1, p_2) \leq L_S(\partial C_{\sigma})/2 = L_S(\sigma) \cosh w = L_S(\sigma) \coth(L_S(\sigma)/2) \leq 3 \) since \( k_1 \geq 6 \) and \( L_S(\sigma) < 2 \arccoth(\cosh k_1) \).

On one hand, since \( f \) is an \((a, b)\)-quasi-geodesic (recall that \( p_2 \in \gamma \)),
\[
(4.5) \quad d_S(f(p_1), f(\gamma)) \leq d_S(f(p_1), f(p_2)) \leq ad_S(p_1, p_2) + b < 3a + b.
\]
On the other hand, taking into account (4.2),
\[
(4.6) \quad d_S(f(p_1), f(\gamma)) \geq d_S(f(\gamma), q) - d_S(q, f(p_1)) \geq 3a + b + c - c = 3a + b.
\]
Obviously (4.6) contradicts (4.5), so such a point \( q \in B \setminus T \) cannot exist and the tube \( T \) mentioned in the statement of the theorem does exist.

(2) The same arguments work for this situation, defining \( k := a(2a + 2b + 2c + t) \).

\[ \square \]

**Lemma 4.3.** Let \( \eta \) be an \((a, b)\)-quasi-geodesic in \( S \) and \( h > 0 \). Then there exists a positive constant \( r_0 \), which just depends on \( a, b, h \), with the following property: if for some \( z_0 \in \eta \) the ball \( B := B_S(z_0, r) \) is simply connected and it is contained in the \((a, b)\)-quasi-isometry (recall that \( p_2 \in \gamma \)),
\[
(4.7) \quad J > \frac{1}{h + \delta} \left( \frac{a^2}{2} (3h + 3\delta + b) + b + 2h + 2\delta \right) + 2 > j_1 > \frac{1}{h + \delta} \left( \frac{a^2}{2} (3h + 3\delta + b) + b + 2h + 2\delta \right).
\]

Let us consider any geodesic \( \gamma_1 \subset B_1 \) extended in both directions from the point \( z_0 \) and a second geodesic \( \gamma_2 \), perpendicular to \( \gamma_1 \) and extended just in one direction from \( z_0 \). Let us fix points \( z_1, z_2, z_3, \ldots \in \gamma_1 \) in one of the directions starting at \( z_0 \) and \( z_{-1}, z_{-2}, z_{-3}, \ldots \in \gamma_1 \) in the opposite direction from \( z_0 \), such that \( d_S(z_0, z_j) = |j|(h + \delta) \) for every \( j \) with \( |j|(h + \delta) < r/2 \). Analogously, choose points \( w_j \in \gamma_2 \) with \( d_S(z_0, w_j) = j(h + \delta) \) for every \( j > 0 \) with \( j(h + \delta) < r/2 \).

Since \( B_1 \) is contained in the \((a, b)\)-neighborhood of \( \eta \), for each of these points \( z_j, w_j \in B_1 \) there exist points \( z_j^*, w_j^* \in \eta \) verifying \( d_S(z_j, z_j^*) \leq h + \delta \) and \( d_S(w_j, w_j^*) \leq h + \delta \).
Let \( t_j, s_j \in I \) be the real values such that \( \eta(t_j) = z_j^* \) and \( \eta(s_j) = w_j^* \) (according to this notation, \( \eta(t_0) = z_0 = z_0^* \)). Then \( |t_j - t_k| \leq a(d_S(z_j^*, z_k^*) + b) \leq a(|j - k|(h + \delta) + 2h + 2\delta + b) \) and, in particular, \( (4.8) \quad |t_j - t_{j+1}| \leq a(3h + 3\delta + b). \)

Note that \( z_j^* \) and \( z_{j+1}^* \) are both in the ball \( B_1): \]

\[
d_{S}(z_{j+1}^*, z_0) \leq d_{S}(z_{j+1}^*, z_j^*) + d_{S}(z_j^*, z_0) \leq h + \delta + J(h + \delta) = (h + \delta)(J + 1) < r/2.
\]

For the defined value \( j_1 \)

\[
|s_{j_1} - t_0| \leq a(d_{S}(w_{j_1}^*, z_0) + b) \leq a(d_{S}(w_{j_1}^*, w_j) + d_{S}(w_j, z_0) + b)
\]

\[
\leq a(h + \delta + j_1(h + \delta) + b).
\]

A similar argument gives,

\[
|t_j - t_0| \geq \frac{1}{a} (d_{S}(z_0, z_j^*) - b) \geq \frac{1}{a} (J(h + \delta) - h - \delta - b).
\]

Using the third inequality in \( (4.7) \) and \( j_1(h + \delta) < a^2/2(3h + 3\delta + b) + b + 4h + 4\delta \), it is easy to check that

\[
\frac{1}{a} (J(h + \delta) - h - \delta - b) > a(h + \delta + b + j_1(h + \delta)).
\]

Therefore, comparing \( (4.9) \) and \( (4.10) \), one obtains \( |t_j - t_0| > |s_{j_1} - t_0| \). Analogously, \( |t_{j+1} - t_0| > |s_{j_1} - t_0| \). Hence, by \( (4.8) \), there exists some \( j_2 \in \mathbb{Z} \) such that

\[
|s_{j_1} - t_{j_2}| \leq \frac{a}{2} (3h + 3\delta + b).
\]

Taking into account the above inequality,

\[
d_{S}(w_{j_1}, z_{j_2}) \leq d_{S}(w_{j_1}^*, z_{j_2}^*) + 2h + 2\delta \leq a|s_{j_1} - t_{j_2}| + b + 2h + 2\delta
\]

\[
\leq \frac{a^2}{2} (3h + 3\delta + b) + b + 2h + 2\delta \quad \text{and}
\]

\[
d_{S}(w_{j_1}, z_{j_2}) \geq d_{S}(w_{j_1}, z_0) = j_1(h + \delta).
\]

Thus,

\[
 j_1(h + \delta) \leq \frac{a^2}{2} (3h + 3\delta + b) + b + 2h + 2\delta,
\]

which contradicts the second inequality in \( (4.7) \). Therefore, \( r \leq 2h(J + 1) \) as claimed. \( \square \)

**Theorem 4.4.** Assume that \( S \) and \( S' \) are genus zero surfaces. Let \( \gamma \) be any geodesic perpendicular to \( \sigma \) contained in \( C_\sigma \) with \( L_S(\gamma) = 2w \). There exist positive constants \( r_0, k_0, k_1, k_2 \), which just depends on \( a, b, c \), so that if \( \gamma_0 := \{ p \in \gamma : d_S(p, \sigma) < w - k_1 \} \) and \( L_S(\sigma) < k_2 \), then there exists an \( (f, \gamma, \gamma_0, h, h + k_0) \)-tube \( T \subset S' \) with \( h := 3a + b + c \).

Furthermore, if \( u_1, u_2 \) are the endpoints of \( \gamma_0 \), and \( g^i \) is any simple geodesic loop with base point \( f(u_i) \) and \( L_{S'}(g^i) = 2(f(u_i)) (i = 1, 2) \), then \( g^2 \) and \( g^2 \) bound a doubly connected set in \( S' \), and for every \( z \in \gamma_0 \), \( \iota(f(z)) \leq r_0 \) and the injectivity radius in \( f(z) \) is attained in the geodesic loop with base point \( f(z) \) freely homotopic to a simple closed geodesic \( \sigma' \) in \( S' \), where \( \sigma' \) only depends on \( \sigma \) and \( f \). Besides, \( f(C_{\sigma}) \) is contained in the \( H_0 \)-neighborhood of the collar \( C_{\sigma'} \) of \( \sigma' \), where \( H_0 := r_0 + ak_1 + b \).

Note that \( \sigma' \) does not depend neither on \( \gamma \) or \( \gamma_0 \).

**Proof.** Let us define \( J \) as the least integer satisfying

\[
J > \frac{1}{a} \left( a^2 \left( \frac{a^2}{2} (3h + b) + 2b + 5h \right) + b + h \right),
\]

\[
r_0 := 2h(J + 1), \quad k_0 := a^2(12h(J + 1) + 1/2 + 3b) + 12h(J + 1) + b + h + 3, \quad k_1 := 2a(3a + 2b + 2c + k_0) \quad \text{and}
\]

\[
k_2 := 2 \arccoth(cosh(k_1 + a(4h(J + 1) + b)/2)).
\]
Since \( L_S(\sigma) < k_2 \), the width \( w \) of the collar \( C_\sigma \) verifies the inequality \( w > k_1 + a(2r_0 + b)/2 \) (see Remark 4.1), and consequently,

\[
d_S(f(u_2), f(u_1)) \geq \frac{1}{a} d_S(u_2, u_1) - b = \frac{2w - 2k_1}{a} - b > 4h(J + 1) = 2r_0.
\]

Therefore, it is possible to choose points \( x_0, x_1, \ldots, x_m \in \gamma_0 \), where \( x_0 = u_1 \), \( x_m = u_2 \) and

\[
2r_0 < d_S(f(x_j), f(x_{j-1})) \leq 4r_0
\]

for \( j = 1, \ldots, m \).

Note that \( h + k_0 > r_0 \). Let us consider any \( r_1 \) with \( h + k_0 > r_1 > r_0 \). By Lemma 4.2 (1), taking \( t = k_0 \), \( V_{r_1+k_0}(f(\gamma_0)) \subset V_h(f(\gamma)) \), and thus Lemma 4.3 gives that the balls \( B_{r_1}(f(x_j), r_1) \) are not simply connected for \( j = 0, 1, \ldots, m \). Therefore, the injectivity radius \( \iota(f(x_j)) \) at the point \( f(x_j) \) is less than \( r_1 \) for every \( r_1 > r_0 \), and then \( \iota(f(u_1)), \iota(f(u_2)) \leq r_0 \). Consequently, there exists a simple geodesic loop \( g_j \) with base point \( f(x_j) \) and \( L_S(g_j) = 2\iota(f(x_j)) \leq 2r_0 \). In light of (4.11), \( g_j \cap g_{j+1} = \emptyset \).

Since \( S' \) is a genus zero surface, \( g_j \) and \( g_j + 1 \) intersect \( S' \), and then \( S'( \{ g_j \cup g_{j+1} \} \cap \gamma_0 = 0 \) has three connected components. Consider the geodesics \( \gamma_j' := \{ f(x_j), f(x_{j+1}) \} \subset S' \) for \( j = 0, \ldots, m - 1 \).

Claim. \( \gamma_j' \cap g_j = \{ f(x_j) \} \) and \( \gamma_j' \cap g_{j+1} = \{ f(x_{j+1}) \} \) for \( j = 0, \ldots, m - 1 \).

Assume the claim holds. Assume that \( g_j \) is not freely homotopic to \( g_{j+1} \) for some \( j \). It will be shown that, in that case, \( B_{r_1+k_0}(f(\gamma_0)) \subset V_h(f(\gamma)) \), contradicting Lemma 4.2 (1). Set \( \eta_j := \{ g_j \cup \gamma_j \cup g_{j+1} \cup (-\gamma_j') \} \). Note that \( L_S(\eta_j) \leq 12r_0 \). By means of a slight modification of \( \eta_j \), one can construct a simple closed curve \( \eta_j' \) freely homotopic to \( \eta_j \), with \( \eta_j' \cap \{ g_j \cup g_{j+1} \} = \emptyset \), \( L_S(\eta_j') \leq 12r_0 \) and \( \mathcal{H}_S(\eta_j, \eta_j') \leq 1 \) (where \( \mathcal{H}_S \) denotes Hausdorff distance) as follows way. Without loss of generality, \( S' \) is a domain contained in \( C \), so take opposite orientation for \( g_j \) and \( g_{j+1} \) and let \( \eta_j \) be an oriented curve. If either \( g_j \) surrounds \( g_{j+1} \) or \( g_{j+1} \) surrounds \( g_j \), choose \( g_j \) contained in the annulus in \( C \) bounded by \( g_j \) and \( g_{j+1} \). Otherwise, choose \( g_j \) contained in the “exterior” connected component of \( S' \\setminus \eta_j \).

Since the curves \( g_j \) and \( \eta_j \) are not trivial, \( g_j \cup \eta_j \) disconnect \( S' \) and \( S' \setminus \{ g_j \cup \eta_j \} \) has four connected components, one of them bounded (with finite diameter) denoted by \( V' \); and other three unbounded (with infinite diameter). Note that \( \partial V' = g_j \cup g_{j+1} \cup \eta_j' \). Since there are three unbounded connected components of \( S' \setminus \{ g_j \cup g_{j+1} \cup \eta_j' \} \), there must exist an unbounded connected component \( U \) with \( f(u), f(v) \notin U \). Note that

\[
\text{diam}_{S'} \partial U \leq \frac{1}{2} \max\{ L_S(g_j), L_S(g_{j+1}), L_S(\eta_j') \} \leq 6r_0 + \frac{1}{2}.
\]

Since \( V \) is contained in the 1-neighborhood of \( \eta_j \) and \( L_S(\eta_j) \leq 12r_0 \), then

\[
\text{diam}_{S'} V \leq 1 + \text{diam}_{S'} \eta_j + 1 \leq 6r_0 + 2.
\]

Assume first that \( f(\gamma) \) intersects \( U \). In this case, \( \text{diam}_{S'} (f(\gamma) \cap \overline{U}) \) is bounded above. Indeed, consider \( \gamma_0 \) to be oriented from \( u_1 \) to \( u_2 \) and consider points \( p' := \{ \tau \in [u_1, u_2] : f(\tau) \in \overline{U} \} \) and \( q' := \{ \tau \in [u_1, u_2] : f(\tau) \in \overline{U} \} \).

Given \( p, q \in \gamma \cap f^{-1}(\overline{U}) \), since \( L_S(\eta_j') \leq 12r_0 + 1 \), \( f(\gamma) \) is a (possibly) discontinuous curve with gaps of amplitude at most \( b \), one deduces

\[
\begin{align*}
d_S(f(p), f(q)) &\leq d_S(p, q) + b + a^2 d_S(f(p'), f(q')) + a^2 b + b \\
&\leq a^2 \text{diam}_{S'} \partial U + 2b + a^2 b + b \\
&\leq a^2 (6r_0 + 1/2 + 3b) + b = k_0 - 6r_0 - h - 3.
\end{align*}
\]

Thus \( \text{diam}_{S'} (f(\gamma) \cap \overline{U}) \leq k_0 - 6r_0 - 2 - h - 1 \). Consequently, if \( z \in f(\gamma) \cap \overline{U} \), then \( d_S(f(x_j), z) \leq \text{diam}_{S'} V + \text{diam}_{S'} (f(\gamma) \cap \overline{U}) \leq k_0 - h - 1 \).

If \( f(\gamma) \) does not intersect \( U \) and \( z \in \overline{U} \), then \( d_S(f(x_j), z) \leq \text{diam}_{S'} V \leq k_0 - h - 1 \).
Therefore, in both cases, since the region $U$ is unbounded, the ball $\overline{B}_S(f(x_j), h_0)$ cannot be contained in $T = V_h(f(\gamma))$, which contradicts Lemma 4.2 (1).

Since it was shown that the closed ball $\overline{B}_S(f(x_j), h + k_0)$ must be contained in $T$, then $\overline{B}_S(f(x_j), k_0)$ must also be contained in $T$.

Therefore, $g_j$ is freely homotopic to $g_{j+1}$ for every $j$. Consequently, the simple geodesic loops $g_0 = g^1$ and $g_m = g^2$ with base points $f(u_1)$ and $f(u_2)$, respectively, are freely homotopic and bound a doubly connected set in $S'$ as claimed.

By taking different sequences of points $\{x_j\}$ one can check that if $z \in \gamma_0$, and $g_z$ is any simple geodesic loop with base point $f(z)$ and $L_S'(g_z) = 2\epsilon(f(z))$, then $g_z$ is freely homotopic to $g^1$ and $\epsilon(f(z)) \leq r_0$.

By Theorem 4.5 below, the map $f$ provides a bijective correspondence from the cusps of $S$ to the cusps of $S'$; hence, there exists a simple closed curve $\sigma'$ freely homotopic to $g^1$ of length $t$. By the Collar Lemma and [6, p.454], the injectivity radius $\tau_0$ at the points in $\partial C_{\sigma'}$ satisfies $\sinh(\tau_0) = \sinh(l/2) > 0$.

For $z \in \gamma_0$, $f(z)$ either belongs to the collar $C_{\sigma'}$, and thus to $V_{\tau_0}(C_{\sigma'})$, or, otherwise, let us define $t := d_S(f(z), \sigma') - w > 0$. Then, $\sinh r_0 \geq \sinh\epsilon(z) = \sinh(l/2) \cosh(w + t)$ and

$$\frac{1}{2} e^t \leq \frac{1}{2} e^{\sinh(l/2) \cosh w} < \frac{1}{2} e^{\sinh(l/2) \cosh(w + t)} \leq \sinh r_0 \leq \frac{1}{2} e^{r_0}.$$ 

Hence, $t < r_0$ and $f(\gamma_0) \subset V_{\tau_0}(C_{\sigma'})$. Given another geodesic $\gamma_0$ perpendicular to $\sigma$, it has been proved that $f(\gamma_0)$ is contained in the $r_0$-neighborhood of $C_{\sigma'}$ for some simple closed curve $\sigma'$ in $S'$; in order to check that $\sigma'$ is $\sigma$'it suffices to repeat the previous argument replacing $f(\gamma_0)$ by any geodesic $\gamma_0$ meeting $\sigma$ with an angle $\pi/2 - \epsilon$ for small $\epsilon > 0$. Therefore, $f(\{p \in S: d_S(p, \sigma) \leq w - k_1\})$ is contained in the $r_0$-neighborhood of $C_{\sigma'}$, and $f(C_{\sigma})$ is contained in the $(a + b + r_0)$-neighborhood of $C_{\sigma'}$.

In order to prove the first part of the claim, assume that there exists a point $\zeta$ in $\gamma_j \cap g_j \setminus \{ f(x_j) \}$ and argue by contradiction.

Denote by $g_j^*$ a subcurve of $g_j$ joining $f(x_j)$ and $\zeta$ and denote by $\gamma_j^*$ the subcurve of $\gamma_j$ joining $f(x_j)$ and $\zeta$. Since $\gamma_j^*$ is a geodesic, $L_S'(<\gamma_j^*>) \leq L_S'(g_j^*)$. Choose $g_j^*$ so that the loop $\Gamma_0 := g_j^* \cup \gamma_j^*$ with base point $f(x_j)$ is non trivial; since $\Gamma_0$ has a corner in $\zeta$, there exists a curve $\Gamma$ freely homotopic to $\Gamma_0$ (thus non trivial) with $L_S'(<\Gamma>) < L_S'(g_j^*) + L_S'(\gamma_j^*) \leq L_S'(g_j)$. This inequality contradicts

$$L_S'(g_j) = 2\epsilon(f(x_j)) = \inf\{L_S'(c) : c \text{ is a loop with base point } f(x_j)\}.$$ 

The proof of the second part of the claim is similar.

The claim, and hence the Theorem, hold. □

**Theorem 4.5.** Assume that $S'$ is a genus zero surface and $f : S \rightarrow S'$ a c-full (a, b)-quasi-isometry. Let $C$ be the 2-collar of a cusp in $S$ with boundary curve $\sigma$ and $\gamma$ an infinite geodesic contained in $C$ perpendicular to $\sigma$. There exist positive constants $H, k, t$, which just depend on $a, b, c$, so that if $\gamma_0 := \{p \in \gamma : d_S(p, \sigma) > k\}$, then there exists an $(f, \gamma; \gamma_0, h, k_0)$-tube $T \subset S'$ with $h := 2a + b + c$. Furthermore, $f(C)$ is contained in the $H$-neighborhood of the 2-collar of a cusp in $S'$.

**Proof.** Define $J$ as the least integer satisfying

$$J > \frac{1}{b} \left( a^2 \left( \frac{a^2}{2} (3h + b) + 2b + 5h \right) + b + h \right).$$ 

Let us consider positive constants $t := a^2(12h(J + 1) + 1/2 + 3b) + 12h(J + 1) + b + h + 3$, $k := a(2a + 2b + 2c + t)$ and $H := ak + b + \log \sinh(2h(J + 1))$.

The statement about the existence of the tube is given by Lemma 4.2 (2), since this lemma holds for any positive value of $t$, and $k$ is defined as in the proof of Lemma 4.2.

Let us choose now two points $u_1, u_2 \in \gamma_0$ such that $d_S(u_1, u_2) > a(4h(J + 1) + b)$, which is always possible since $\gamma_0$ is infinite. Notice that

$$d_S(f(u_2), f(u_1)) \geq \frac{1}{a} d_S(u_2, u_1) - b > 4h(J + 1).$$
Let us define the constant $C := a^2(12h(J + 1) + 1/2 + 2b) + 12b(J + 1) + ab + b + b + 3$. From this point on, the conclusion of this theorem can be obtained repeating the reasoning offered in the proof of Theorem 4.4 with $C$ playing the role of $k_0$. However, since now the geodesic $\gamma_0$ is infinite, the distance between $u_1$ and $u_2$ can be arbitrarily large. Then, all the geodesic loops $\{g_j\}$ with base point $f(x_j)$, for any $x_j$ located on $\gamma_0$ are homotopic and, besides, $L_S(g_j) \leq 4h(J + 1)$. It means that $f(\mathcal{E})$ is actually contained in a neighborhood of a cusp in $S'$, since $f(\mathcal{E})$ is not a bounded set.

Next, it will be shown that $f(\mathcal{E})$ is inside the $H$-neighborhood of the 2-collar of a cusp in $S'$. Let us choose any of the geodesic loops $g_j$ mentioned above, and let us assume that it lies out of the 2-collar of the corresponding cusp in $S'$. As usual, consider a fundamental domain for $S'$ in the upper halfplane $H$ contained in $\{z \in H : 0 \leq \Re z \leq 1\}$ and such that $\{z \in H : 0 \leq \Re z \leq 1, 3z \geq 1/2\}$ corresponds to the 2-collar of this cusp in $S'$. Let us represent $g_j$ in the upper half-plane by means of a geodesic with endpoints $i\alpha$ and $i\alpha + 1$. If $\alpha \geq 1/2$, then $g_j$ is in the 2-collar in $S'$. Hence, $\alpha < 1/2$. If $d$ is the actual length of $g_j$, then $\sinh(d/2) = 1/(2\alpha)$. Taking into account that $d = L_S(g_j) \leq 4h(J + 1)$, then $1/(2\alpha) = \sinh(d/2) \leq \sinh(2h(J + 1))$, and

$$d_H(i\alpha, i/2) = \log \frac{1}{2\alpha} \leq \log \sinh(2h(J + 1)) =: H_1,$$

which means that $x_j$ is in the $H_1$-neighborhood of the boundary curve of the 2-collar in $S'$. Hence, in any case, $x_j$ is in the $H_1$-neighborhood of the 2-collar in $S'$. Therefore, $f(\{p \in \mathcal{E} : d_S(p, \sigma) > k\})$ is contained in the $H_1$-neighborhood of the 2-collar in $S'$, and $f(\mathcal{E})$ is contained in the $(H_1 + ak + b)$-neighborhood of the 2-collar of a cusp in $S'$.

Lemma 4.6. Fix two positive constants $d_1$ and $t_0$. Let $S$ be a non-exceptional Riemann surface, $s$ be a simple closed geodesic with $S\setminus \sigma$ non-connected, and $x, y$ points in $S$ such that $d_S(x, y) \geq d_1$ and the geodesic loops $g_x, g_y$ with base points $x, y$, respectively, freely homotopic to $s$, verify $L_S(g_x), L_S(g_y) \leq 2t_0$. Let $\sigma_x$ (respectively, $\sigma_y$) be the set of points in the connected component of $S\setminus \sigma$ containing $x$ (respectively, $y$) which are at distance $d_S(x, \sigma)$ (respectively, $d_S(y, \sigma)$) from $\sigma$; denote by $C$ the domain in $S$ bounded by $\sigma_x$ and $\sigma_y$, and by $C_0$ the set of points in $C$ at distance greater or equal than $d_2$ from $\partial C = \sigma_x \cup \sigma_y$, with

$$0 < d_2 < \arccosh \frac{2 \cosh^2(d_1/2)}{\sqrt{4 \cosh^2(d_1/2) + \sinh^2 t_0}}.$$

Then $C_0$ is non empty and $\sinh(\zeta) < 2 e^{-d_2} \sinh t_0$ for every $\zeta \in C_0$.

Remark 4.7. An elementary computation gives that if $d_1 = \max\{2t_0, 2d_2 + \log 20\}$, then (4.13) holds.

Proof. Define $l := L_S(\sigma)$ and $x_0$ (respectively, $y_0$) the point in $\sigma$ with $d_S(x, \sigma) = d_S(x, x_0)$ (respectively, $d_S(y, \sigma) = d_S(y, y_0)$). It is clear that the maximum of the injectivity radius is attained when $S$ is an annulus with simple closed geodesic $s$, $d_S(x, y) = d_1$, $L_S(g_x), L_S(g_y) = 2t_0$, and $x, y$ are antipodal points with respect to $\sigma$, i.e., $d_S(x, \sigma) = d_S(y, \sigma)$ and $d_S(x, y) = l/2$. In this case, defining $u := d_S(x, \sigma) = d_S(y, \sigma)$, it is well-known (see, e.g., [6, p.454]) that $\cosh(d_1/2) = \cosh u \cosh(l/4)$ and $t_0 = \sinh(l/2) \cosh u$. Hence,

$$\sinh(l/4) = \frac{\sinh(l/2)}{2 \cosh(l/4)} = \frac{\sinh t_0}{2 \cosh(d_1/2)}, \quad \cosh u = \frac{\cosh(d_1/2)}{\cosh(l/4)} = \frac{2 \cosh^2(d_1/2)}{\sqrt{4 \cosh^4(d_1/2) + \sinh^4 t_0}}.$$

Therefore, take $0 < d_2 < d_S(x, \sigma) = d_S(y, \sigma)$ and $C_0$ is not the empty set. If $u \in C_0$, denote by $\psi_u$ the geodesic loop with base point $u$ freely homotopic to $s$. For any $z \in \partial C_0$, it is well-known (see, e.g., [6, p.454]) that

$$\sinh(L_S(\psi_z)/2) = \sinh(l/2) \cosh(u - d_2) \leq e^{u-d_2} \sinh(l/2) < 2 e^{-d_2} \sinh(l/2) \cosh u = 2 e^{-d_2} \sinh t_0.$$

Hence, for any $z \in C_0$, $\sinh(L_S(\psi_z)/2) < 2 e^{-d_2} \sinh t_0$ and, consequently, $\sinh(\zeta) < 2 e^{-d_2} \sinh t_0$. 

Lemma 4.8. Let $S$ be a non-exceptional Riemann surface, and $z \in S$. If $\zeta(z) < \arcsinh 1$, then the shortest geodesic loop $\eta$ with base point $z$ is contained either in the 2-collar of a cusp or in the collar $C_0$ of a simple closed geodesic $s$. 

Proof. Given any point \( p \) on the boundary of the 2-collar of a cusp, or on the boundary of the collar of a simple closed geodesic then \( \iota(p) \geq \text{arcsinh } 1 \) by the Collar Lemma.

Therefore \( z \) must lie inside the cusp or collar. Lifting its shortest geodesic loop to the universal covering shows this will also be the case for its geodesic loop.

**Lemma 4.9.** Let \( S \) be a non-exceptional Riemann surface, \( \sigma \) be a simple closed geodesic in \( S \) and \( z \) a point in the collar \( C_\sigma \). Then \( d_S(z, \partial C_\sigma) \geq \log(1/\sinh \iota(z)) \).

Proof. If \( \iota(z) \geq \text{arcsinh } 1 \), then \( d_S(z, \partial C_\sigma) \geq 0 \geq \log(1/\sinh \iota(z)) \). Assume now that \( \iota(z) < \text{arcsinh } 1 \). By Lemma 4.8 the shortest geodesic loop \( \eta \) with base point \( z \) is contained in \( C_\sigma \). Note that if \( d := d_S(z, \sigma) \), then \( d_S(z, \partial C_\sigma) = w - d \). Let \( l := L_S(\sigma) \); by the Collar Lemma and [6, p.454]:

\[
\sinh \iota(z) = \sinh(l/2) \cosh d = \frac{\cosh d}{\sinh w} \geq e^{d-w},
\]

which implies the result.

**Lemma 4.10.** Let \( S \) be a non-exceptional genus zero Riemann surface, \( I \) and \( h \) two positive constants, \( \sigma \) a simple closed geodesic with \( L_S(\sigma) \leq 2I \), and \( C_\sigma^h \) the \( h \)-neighborhood of \( C_\sigma \). Denote by \( S_1 \) a connected component of \( S \setminus \sigma \), and by \( \alpha_1 \) the set of closed curves in \( \partial C_\sigma^h \setminus S_1 \). If \( p, q \in \alpha_1 \), then \( d_S(p, q) \leq e^h I \cosh I \).

Proof. Without loss of generality, assume that \( S \) is an annulus and \( \sigma \) is the simple closed geodesic in \( S \). Define \( l := L_S(\sigma) \) and \( L := L_S(\alpha_1) \). Since \( l/2 \leq I \) and \( g(x) = x \cosh x \) is an increasing function for \( x > 0 \),

\[
d_S(p, q) \leq L/2 = (l/2) \cosh w \frac{\cosh(w + h)}{\cosh w} < (l/2) \coth(l/2) \cosh(w + h) \cosh w \leq e^h I \cosh I.
\]

Finally, the following two lemmas are easy to check.

**Lemma 4.11.** Let \( g : [\alpha, \beta] \to \mathbb{R} \) be an \((a, b)\)-quasi-isometric embedding with \( g(\beta) > g(\alpha) \). If \( x, y \in [\alpha, \beta] \) and \( y > x + (a + 1)b \), then \( g(y) > g(x) \).

**Lemma 4.12.** Let \( g : [0, \infty) \to [0, \infty) \) be an \((a, b)\)-quasi-isometric embedding. If \( x, y \geq 0 \) and \( y > x + (a + 1)b \), then \( g(y) > g(x) \).

5. Stability of the injectivity radius under quasi-isometries

Recall the notation \( C_\alpha \) and \( \mathcal{C}_\alpha \) for collars of simple closed geodesic and cusps respectively. Also denote by \( \mathcal{C}_\alpha \) the \( H \)-neighborhood of the 2-collar of a cusp with boundary \( \alpha \) (now \( \alpha \) can be a union of closed curves).

**Theorem 5.1.** Let \( S \) and \( S' \) be non-exceptional genus zero Riemann surfaces and let \( f : S \to S' \) be a \( c \)-full \((a, b)\)-quasi-isometry. For each \( \varepsilon' > 0 \) there exists \( \varepsilon > 0 \) which just depends on \( \varepsilon', a, b, c \), such that if \( \iota(z) < \varepsilon \) then \( l(f(z)) < \varepsilon' \). Moreover, given \( \varepsilon_1 > 0 \), \( \varepsilon \) can be taken so that \( \varepsilon < \varepsilon_1 \).

Proof. Without loss of generality assume that \( 0 < \varepsilon' \). Fix \( z \in S \) with \( \iota(z) < \text{arcsinh } 1 \).

The proof takes advantage of the relation between \( \iota(z) \) and the distance from \( z \) to the boundary of the collar of a cusp when \( z \) is in the interior of the collar of a cusp, or the distance from \( z \) to the boundary of the collar of a simple closed geodesic when \( z \) is in the collar.

Assume first that the shortest geodesic loop based on \( z \) is freely homotopic to a cusp in \( S \). Let \( z \) belong to the interior of the 2-collar of this cusp, \( \mathcal{C}_\alpha \), where \( \alpha \) is its boundary curve. In this setting

\[
\sinh \iota(z) = e^{-d_S(z, \alpha)} < 1.
\]

For every \( u \in \mathcal{C}_\alpha \) define \( W_\alpha(u) := d_S(u, \alpha) \). Then for any two points \( u, v \in \mathcal{C}_\alpha \),

\[
|W_\alpha(v) - W_\alpha(u)| \leq d_S(v, u) \leq |W_\alpha(v) - W_\alpha(u)| + 1,
\]

(5.14)
By Theorem 4.5, \( f(\mathcal{C}_a) \) is contained in \( \mathcal{C}_a' \), the \( H \)-neighborhood of the 2-collar of a cusp in \( S' \) (now \( \alpha' := \partial \mathcal{C}_a' \) can be a union of closed curves). Let us define \( W_{\alpha'}(p) := d_{S'}(p, \alpha') \) for every \( p \in \mathcal{C}_a' \). Then,

\[
|W_{\alpha'}(f(v)) - W_{\alpha'}(f(u))| \leq d_{S'}(f(v), f(u)) \leq |W_{\alpha'}(f(v)) - W_{\alpha'}(f(u))| + 1 + 2H,
\]

for any two points \( u, v \in \mathcal{C}_a' \).

By (5.14) and (5.15),

\[
|W_{\alpha'}(f(v)) - W_{\alpha'}(f(u))| \leq d_{S'}(f(v), f(u)) \leq ad_{S}(v, u) + b \leq a|W_{\alpha'}(v) - W_{\alpha'}(u)| + 1 + b
\]

\[
= a|W_{\alpha'}(v) - W_{\alpha'}(u)| + a + b,
\]

(5.16)

\[
|W_{\alpha'}(f(v)) - W_{\alpha'}(f(u))| \geq d_{S'}(f(v), f(u)) - 1 - 2H \geq \frac{1}{a} d_{S}(v, u) - b - 1 - 2H
\]

\[
\geq \frac{1}{a} |W_{\alpha'}(v) - W_{\alpha'}(u)| - 1 - b - 2H,
\]

for any two points \( u, v \in \mathcal{C}_a' \). Therefore, (5.16) shows that there is an \( (a, a + b + 2H) \) quasi-isometric embedding defined from \([0, \infty) \) to \([0, \infty) \) that relates \( W_{\alpha'}(u) \) with \( W_{\alpha'}(f(u)) \), for every \( u \in \mathcal{C}_a' \).

Let \( z_0 \) be the point in \( \sigma \) so that \( d_{S}(z, \alpha) = d_{S}(z, z_0) \). Then

\[
W_{\alpha}(z) - W_{\alpha}(z_0) = W_{\alpha}(z) - \log \frac{1}{\sinh \epsilon(z)} > \log \frac{1}{\sinh \epsilon}.
\]

Choosing \( \epsilon \) so that \( \sinh \epsilon < e^{-(a+1)(a+b+2H)} \),

\[
W_{\alpha}(z) - W_{\alpha}(z_0) > (a + 1)(a + b + 2H).
\]

By Lemma 4.12, \( W_{\alpha'}(f(z)) > W_{\alpha'}(f(z_0)) \), and thus (5.16) and (5.17) give

\[
W_{\alpha'}(f(z)) \geq W_{\alpha'}(f(z_0)) \geq \left| W_{\alpha'}(f(z)) - W_{\alpha'}(f(z_0)) \right|
\]

\[
\geq \frac{1}{a} |W_{\alpha}(z) - W_{\alpha}(z_0)| - 1 - b - 2H \geq \frac{1}{a} \log \frac{1}{\sinh \epsilon} - 1 - b - 2H > \log \frac{1}{\sinh \epsilon} + H \geq H,
\]

since \( 0 < \epsilon' \leq \arcsinh 1 \), and if \( \epsilon \) is taken to be

\[
\sinh \epsilon < \min \left\{ \left( \sinh \epsilon' \right)^a e^{-a(3H+1+b)}, e^{-(a+1)(a+b+2H)} \right\}.
\]

Since \( W_{\alpha'}(f(z)) > H \), \( f(z) \) is in the 2-collar of the cusp, and thus \( W_{\alpha'}(f(z)) = - \log \sinh \epsilon(f(z)) + H \). Hence \( \epsilon(f(z)) < \epsilon' \).

Assume now that the shortest loop based on \( z \) is freely homotopic to a simple closed geodesic \( \sigma \); then \( z \) belongs to the interior of the collar \( C_a \) of width \( w \). Let us consider the geodesics \( \gamma, \gamma_0 \) and the constants \( r_0, k_0, k_1, k_2, H_0 \) as in Theorem 4.4. If we require \( \epsilon \leq k_2/2 \), then \( \epsilon := L_\sigma(\gamma) \leq 2\epsilon(z) < k_2 \). By the Collar Lemma and \([6, p.45]\) we have that the length \( \ell_0 \) of the geodesic loop freely homotopic to \( \sigma \) and based in any point in \( \partial C_\sigma \) satisfies \( \sinh \ell_0 = \sinh(l/2) \cosh w = \cosh(l/2) \leq \cosh(k_2/2) = k_3 \); thus \( \epsilon(u) < \arcsinh k_3 \) for every \( u \in C_\sigma \).

Denote by \( \alpha_1, \alpha_2 \) the simple closed curves in \( \partial C_\sigma \); then \( L_\sigma(\alpha_j) = \ell \cos \theta < 2 \sinh(l/2) \cos \theta = 2 \cos(l/2) < 2 \cos(k_2/2) = 2k_3 \) for \( j = 1, 2 \). Define \( W_{\alpha_j}(u) := d_S(u, \alpha_j) \) for every \( u \in C_\sigma \) and \( j = 1, 2 \).

Since \( S \) is a genus zero surface,

\[
|W_{\alpha_j}(v) - W_{\alpha_j}(u)| \leq d_S(v, u) \leq |W_{\alpha_j}(v) - W_{\alpha_j}(u)| + k_3,
\]

for any two points \( u, v \in C_\sigma \).

By Theorem 4.4, \( f(C_\sigma) \) is contained in the \( H_0 \)-neighborhood of the collar \( C_\sigma' \) of a simple closed geodesic \( \sigma' \) in \( S' \).

Denote by \( \Psi_u \) the geodesic loop with base point \( f(u) \) freely homotopic to \( \sigma' \). Let \( u \in C_\sigma \), then

\[
L_{\sigma'}(\Psi_u) \leq 2(r_0 + ak_1 + b) := 2r_0' \equiv 2r_0
\]

since if \( d_S(u, \partial C_\sigma) \geq k_1 \), Theorem 4.4 gives \( L_{\sigma'}(\Psi_u) = 2\ell(f(u)) \leq 2r_0 \).
\[ S' \text{ is also a genus zero surface, therefore } S' \setminus \sigma' \text{ has two connected components } S'_1, S'_2. \text{ Then, } f(C_\sigma) \text{ intersects either both of them or only one of them. In the former case, define } r_j := \sup \{ d_{S'}(f(u), \sigma')/u \in C_\sigma, f(u) \in S'_j \} \text{ and } \alpha'_j := \{ v \in S'_j : d_{S'}(v, \sigma') = r_j \} \text{ for } j = 1, 2. \\
In the latter case, define } r_1 := \inf \{ d_{S'}(f(u), \sigma')/u \in C_\sigma \}, r_2 := \sup \{ d_{S'}(f(u), \sigma')/u \in C_\sigma \}, \text{ and } \alpha'_j := \{ v \in S'_j : d_{S'}(v, \sigma') = r_j \} \text{ for } j = 1, 2 \text{ where } i \text{ so that } S'_1 \cap f(C_\sigma) \neq \emptyset. \\
Let } C'_\sigma \text{, the domain in } S' \text{ bounded by } \alpha'_1 \text{ and } \alpha'_2. \text{ For any } p \in C'_\sigma, \text{ define } W_{\alpha'_j}(p) := d_{S'}(p, \alpha'_j), j = 1, 2. \]

By Lemma 4.10, for any } u, v \in C_\sigma, 
\begin{equation}
|W_{\alpha'_j}(f(v)) - W_{\alpha'_j}(f(u))| \leq d_{S'}(f(v), f(u)) \leq |W_{\alpha'_j}(f(v)) - W_{\alpha'_j}(f(u))| + e^{H_{\sigma}r_0} \coth r_0,
\end{equation}

By virtue of (5.18) and (5.19), for any } u, v \in C_\sigma, 
\begin{equation}
|W_{\alpha'_j}(f(v)) - W_{\alpha'_j}(f(u))| \leq d_{S'}(f(v), f(u)) \leq ad_{S'}(v, u) + b \leq a(|W_{\alpha_j}(v) - W_{\alpha_j}(u)| + k_3) + b = a|W_{\alpha_j}(v) - W_{\alpha_j}(u)| + ak_3 + b,
\end{equation}

If } \gamma \text{ is a geodesic orthogonal to } \sigma, \text{ and setting } k_4 := e^{H_{\sigma}r_0} \coth r_0, \text{ then (5.20) shows that there are two } (a, ak_3 + b + k_4)-\text{quasi-isometric embeddings defined from } [0, 2w] \text{ to } R \text{ that relate } W_{\alpha_j}(u) \text{ with } W_{\alpha'_j}(f(u)), \text{ for every } u \in \gamma \text{ and } j = 1, 2.

Let } z \in \gamma \text{ and let } z_j := \gamma \cap \alpha_j \text{ for } j = 1, 2. \text{ By Lemma 4.9, } 
W_{\alpha_j}(z) - W_{\alpha_j}(z_j) = W_{\alpha_j}(z) \geq d_S(z, \partial C_\sigma) \geq \log \frac{1}{\sinh (z)} > \log \frac{1}{\sinh \varepsilon} \geq (a + 1)(ak_3 + b + k_4),

if } \varepsilon \text{ is taken to be } \varepsilon \leq \arcsinh e^{-(a+1)(ak_3+b+k_4)}.

Without loss of generality, label } \alpha'_1 \text{ and } \alpha'_2 \text{ so that } f(z_1) \text{ is closest to } \alpha'_i \text{ for } i = 1, 2.

Therefore, by Lemma 4.11 together with (5.20), 
\begin{equation}
W_{\alpha'_j}(f(z)) \geq |W_{\alpha'_j}(f(z)) - W_{\alpha'_j}(f((z_j))| \geq \frac{1}{a} |W_{\alpha_j}(z) - W_{\alpha_j}(z_j)| - k_4 - b = \frac{1}{a} W_{\alpha_j}(z) - k_4 - b.
\end{equation}

By Lemma 4.9 
\begin{equation}
d_S(z, \partial C_\sigma) \geq \log \frac{1}{\sinh (z)} > \log \frac{1}{\sinh \varepsilon}
\end{equation}

If } \varepsilon \text{ is taken to be so that } 
\begin{equation}
\log \frac{1}{\sinh \varepsilon} \geq a \left(1 + b + k_4 + \log \frac{2 \sinh r_0}{\sinh \varepsilon} \right) =: k_4^*,
\end{equation}

Lemma 4.9 gives, together with the quasi-isometric embedding, 
\begin{equation}
W_{\alpha'_j}(f(z)) \geq \frac{1}{a} W_{\alpha_j}(z) - k_4 - b \geq \frac{1}{a} d_S(z, \partial C_\sigma) - k_4 - b > \log \frac{2 \sinh r_0}{\sinh \varepsilon^2}.
\end{equation}

Set } d_2 := 1 + \log \frac{2 \sinh r_0}{\sinh \varepsilon^2} \text{ and } d_1 := d_{S'}(f(z_1), f(z_2)). \text{ In order to apply Remark 4.7, } d_1 \text{ should satisfy } d_1 > \max \{2r_0', 2d_2 + \log 20 \}, \text{ since } \iota(f(z_j)) \leq r_0'. \text{ Using } f \text{ is an } (a, b)-\text{quasi-isometry,}
\begin{equation}
d_1 \geq \frac{1}{a} d_S(z_1, z_2) - b = \frac{2w}{a} - b.
\end{equation}

By the Collar Lemma together with } \iota(z) < \varepsilon, \text{ the width } w \text{ satisfies } \cosh w > \coth \varepsilon. \text{ Therefore, it suffices to choose } \varepsilon \text{ to be so that } \cosh \varepsilon \geq \cosh \left( \frac{w}{2} (b + \max \{2r_0', 2d_2 + \log 20 \}) \right).

Hence, by Lemma 4.6, } \sinh \iota(f(z)) < 2e^{-d_2} \sinh r_0' \text{ if } \iota(z) < \varepsilon \text{ where } \varepsilon \text{ must satisfy all the above restrictions, namely:}
\begin{equation}
0 < \varepsilon \leq \min \left\{ \frac{k_2}{2}, \arcsinh e^{-(a+1)(ak_3+b+k_4)}, \arcsinh e^{-k_4^*}, \arccoth \cosh \frac{a(b + \max \{2r_0', 2d_2 + \log 20 \})}{2} \right\}.
\end{equation}
Note that the constant $\varepsilon$ in Theorem 5.1 does not depend on $z, f, S, S'$. 

**Theorem 5.2.** Let $S$ and $S'$ be non-exceptional genus zero Riemann surfaces and $f : S \to S'$ a $c$-full $(a,b)$-quasi-isometry. For each $\varepsilon > 0$ there exists $\varepsilon' > 0$ which just depends on $\varepsilon, a, b, c$, such that if $\iota(z) \geq \varepsilon$ then $\iota(f(z)) \geq \varepsilon'$.

**Proof.** For each fixed $z \in S$ let us define a function $F_z : S' \to S$ as follows: $F_z(f(z)) := z$; for each $y \in f(S) \setminus \{f(z)\}$ fix any $x \in f^{-1}(y)$ and define $F_z(y) := x$; finally, for each $y \in S' \setminus f(S)$ choose any $x \in S$ with $d_S(f(x), y) \leq c$ and define $F_z(y) := x$. It is easy to check that $F_z$ is an $ab$-full $(a, a(b + 2c))$-quasi-isometry.

Consequently, by Theorem 5.1, for each $\varepsilon > 0$ there exists $\varepsilon' > 0$, which just depends on $\varepsilon, a, b, c$, such that if $\iota(p) < \varepsilon$ then $\iota(F_z(p)) < \varepsilon$. In particular, if $\iota(f(z)) < \varepsilon'$ then $\iota(z) = \iota(F_z(f(z))) < \varepsilon$. Since $\varepsilon'$ does not depend on $z, f$ or $F_z$, then $\iota(z) < \varepsilon$ for every $z \in S$ with $\iota(f(z)) < \varepsilon'$.

6. PROOF OF THEOREM 1.1

This section is devoted to the proof of Theorem 1.1, which follows Kanai’s approach. In Kanai’s results it is essential that both $\iota(S)$ and $\iota(S')$ are positive; these conditions will be avoided due to Theorems 5.1 and 5.2 and the thick-thin decomposition of Riemann surfaces given by Margulis Lemma (see, e.g., [2, p.107]). Concretely, for any $\varepsilon < \text{arsinh} 1$ any Riemann surface, $S$, can be partitioned into a thick part, $S_\varepsilon := \{z \in S : \iota(z) > \varepsilon\}$, and a thin part, $S \setminus S_\varepsilon$, whose components are either collars of cusps or collars of closed geodesics of length less than or equal to $2\varepsilon$ (see Lemma 4.8).

In order to prove Theorem 1.1, it will be shown that it suffices to consider the thick parts of $S$ and $S'$ for some particular choices of $\varepsilon$ and $\varepsilon'$, so that Kanai’s insight can be brought to $S_\varepsilon$ and $S'_\varepsilon$, if we avoid the (possible) contribution to the $\text{LII}$ given by $\partial S_\varepsilon$ and $\partial S'_\varepsilon$.

We will need the following improvement of Theorems 5.1 and 5.2.

**Lemma 6.1.** Let $S$ and $S'$ be non-exceptional genus zero Riemann surfaces, and $f : S \to S'$ be a $c$-full $(a,b)$-quasi-isometry. Then, given $0 < \varepsilon, \varepsilon_1 < \text{arsinh} 1$, there exist $0 < \varepsilon', \varepsilon < \varepsilon_1$, which just depend on $\varepsilon, \varepsilon_1, a, b, c$, so that

$$f(S_\varepsilon) \subset S'_{\varepsilon'} \subset V_\varepsilon(f(S_\varepsilon)).$$

**Proof.** Theorem 5.2 asserts that given $\varepsilon$ there exists $\varepsilon'$ so that the first inclusion holds. For the second one, given $\varepsilon'$ there exists $\varepsilon''$ such that $S' \setminus S'_{\varepsilon''} \supset V_\varepsilon(S' \setminus S'_{\varepsilon''})$ by Lemma 3.2. Let $z' \in S'_{\varepsilon''}$; then $V_\varepsilon(z') \subset S'_{\varepsilon'}$ and since $f$ is $c$-full, there exists $x' \in V_\varepsilon(z')$ so that $x' = f(x)$ for some $x \in S_\varepsilon$ where $\varepsilon$ is given by $\varepsilon'$ in Theorem 5.1. Therefore $z' \in V_\varepsilon(f(S_\varepsilon))$. Since $S_t$ becomes larger as $t > 0$ decreases, one can obtain $0 < \varepsilon', \varepsilon < \varepsilon_1$. □

As a first goal it is going to be proved the $\text{LII}$ intrinsic to a bordered surface, $S_\varepsilon$ contained in $S$; note that $S_\varepsilon$ is not necessarily connected. To this end, we define below the “thick” boundary of a subset of $S$ as its intrinsic boundary in $S_\varepsilon$, and the “intrinsic” $\text{LII}$ that will refer to as $\text{LII}_\varepsilon$.

**Definition 6.2.** Given a non-exceptional Riemann surface $S_\varepsilon$, $\varepsilon > 0$ and a domain $\Omega$ in $S_\varepsilon$, define

$$\partial_\varepsilon \Omega := \partial \Omega \cap S_\varepsilon = \partial \Omega \setminus \partial S_\varepsilon.$$

**Remark 6.3.** If $\gamma$ is a non-trivial simple closed curve, $\gamma \subset \partial_\varepsilon \Omega$, then $L_\varepsilon(\gamma) > 2\varepsilon$.

**Definition 6.4.** $S_\varepsilon$ is said to satisfy the $\varepsilon$-linear isoperimetric inequality, $\text{LII}_\varepsilon$, if there exists a positive constant $c$, such that if $\Omega$ is a relatively compact domain in $S_\varepsilon$ with smooth boundary, then

$$(6.21) \quad A_\varepsilon(\Omega) \leq c L_\varepsilon(\partial_\varepsilon \Omega).$$
A reduction is that it suffices to prove $LII_c$ for intrinsic geodesic domains in $S_c$. A domain $\Omega \subset S$ is said to be a geodesic domain if $\partial \Omega$ is a finite number of simple closed geodesics, and $A_S(\Omega)$ is finite. Note that $\Omega$ does not need to be relatively compact for it could contain a finite number of cusps. From this point of view, the boundary of a cusp will be considered as an improper geodesic of zero length. An intrinsic geodesic domain is a geodesic domain intrinsic to $S_c$, i.e., the intersection of a geodesic domain in $S$ with $S_c$.

Let us denote by $c_1(S_c)$ the sharp $\varepsilon$-linear isoperimetric constant of $S_c$ and by $c_{1,g}(S_c)$ the sharp $\varepsilon$-linear isoperimetric constant of $S_c$ for intrinsic geodesic domains.

**Lemma 6.5.** Let $S$ be a non-exceptional Riemann surface and $\varepsilon \geq 0$ so that $\varepsilon < \text{arsinh} 1$. Then, 
$S_c$ has $LII_c$ $\iff$ $S_c$ has $LII_c$ for intrinsic geodesic domains in $S_c$.

In fact, $c_{1,g}(S_c) \leq c_1(S_c) \leq c_{1,g}(S_c) + 2$.

Note that this lemma also holds for $S$, corresponding to the case $\varepsilon = 0$; Lemma 6.5 with $\varepsilon = 0$ was proved in [13, Lemma 1.2] and improved in [21, Theorem 7].

**Proof.** The first inequality is direct. For the second one, the Collar Lemma and Bers Theorem (see [4]) give $LII_c$ with constant 2 for simply connected and doubly connected domains. It is well known that these domains satisfy $LII$ with constant 1 (see [13, Lemma 1.2] and [21, Theorem 7]). For other domains, $\Omega \subset S_c$, write $\Omega = \bigcup_{j=1}^{n} \gamma_j$, where each $\gamma_j$ can be assumed to be a non-trivial simple closed curve and $n \geq 3$. Consider $\tilde{\Omega}$, the intrinsic geodesic domain in $S_c$ bounded by $\bigcup_{j=1}^{n} \beta_j$ where $\beta_j$ is the intrinsic geodesic in $S_c \cup \partial S_c$ homotopic to $\gamma_j$. Then $L_S(\partial_i \tilde{\Omega}) \leq L_S(\partial_i \Omega)$ and $A_S(\Omega) \leq A_S(\tilde{\Omega}) + A_S(\Omega \setminus \tilde{\Omega})$ where $\Omega \setminus \tilde{\Omega}$ is a disjoint union of doubly connected domains, each component bounded by a pair $\beta_j$ and $\gamma_j$, or simply connected domains bounded by subsets of $\beta_j$ and $\gamma_j$ with the same endpoints. Since $L_S(\beta_j) \leq L_S(\gamma_j)$, applying the $LII$ for simply and doubly connected domains to each component of $\Omega \setminus \tilde{\Omega}$ one gets $A_S(\Omega \setminus \tilde{\Omega}) \leq 2L_S(\partial_i \tilde{\Omega})$ and thus

$$A_S(\Omega) \leq A_S(\tilde{\Omega}) + 2L_S(\partial_i \tilde{\Omega}) \leq c_{1,g}(S_c)L_S(\partial_i \Omega) + 2L_S(\partial_i \tilde{\Omega}),$$

and then $c_1(S_c) \leq c_{1,g}(S_c) + 2$. This inequality and the first one prove the lemma. \qed

Finally the $LII$ in $S$ can be deduced from the $LII_c$ in $S_c$.

**Proposition 6.6.** Let $S$ be a non-exceptional Riemann surface. Then there exists a universal positive constant $\varepsilon_0 \leq \text{arsinh} 1$ verifying the following properties:

1. If $S_c$ has $LII_c$ for some $0 < \varepsilon < \varepsilon_0$, then $S$ has $LII$. Moreover, $c_1(S) \leq 2c_{1,g}(S_c) + 2$.
2. If $S$ has $LII$, then $S_c$ has $LII_c$ for every $0 < \varepsilon < \min \{\varepsilon_0, (12c_{1,g}(S))^{-1}\}$. Moreover, $c_1(S_c) \leq \frac{2\pi c_{1,g}(S)}{2\pi - 1} + 2$.

**Proof.** By the Collar Lemma, there exists a positive constant $\varepsilon_0 \leq \text{arsinh} 1$ so that if $0 < \varepsilon < \varepsilon_0$, then $A_S(C \setminus S_c) \leq A_S(C \cap S_c)$ for all $C$ collars in $S$, and $L_S(\eta) \leq 3\varepsilon$ for every closed curve $\eta \subset \partial S_c$.

In order to prove the first item, consider any fixed geodesic domain $\Omega \subset S$. Then

$$\Omega \cap S_c = \Omega_1 \cup \cdots \cup \Omega_m$$

with $\{\Omega_k\}$ disjoint intrinsic geodesic domains in $S_c$. Since $0 < \varepsilon < \varepsilon_0$,

$$A_S(\Omega) = A_S(\Omega \cap S_c) + A_S(\Omega \setminus S_c) \leq 2A_S(\Omega \cap S_c)$$

$$= 2 \sum_k A_S(\Omega_k) \leq 2c_{1,g}(S_c) \sum_k L_S(\partial_i \Omega_k) = 2c_{1,g}(S_c)L_S(\partial \Omega).$$

Then $c_1(S) \leq 2c_{1,g}(S_c)$ and Lemma 6.5 gives the first item.

By Lemma 6.5, the proof of the second item will follow if it is shown that when $S$ satisfies the $LII$ then $S_c$ satisfies the $LII_c$ for intrinsic geodesic domains. It will first be shown, that as a consequence of the $LII$ in $S$, for any geodesic domain $\Omega$ in $S$, the length of the short curves of its boundary is controlled by the length of the long curves; concretely, $L_S(\partial_i \Omega) \geq (2\pi - 1)L_S(\partial \Omega \setminus \partial_i \Omega)$. 

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Let \( \Omega \) a geodesic domain in \( S \) with \( \partial \Omega = \bigcup_{j=1}^{n} \beta_j \) (each \( \beta_j \) is either a simple closed geodesic or a cusp and \( n \geq 3 \)) and define \( J := \{ j : L_S(\beta_j) < (2c_{1,g}(S))^{-1} \} \) where \( c_{1,g}(S) \) is the LII constant in \( S \) for geodesic domains. Then, if \( g \) denotes the genus of \( \tilde{\Omega} \), by Gauss-Bonnet Theorem, the LII can be written as
\[
c_{1,g}(S) \left( \sum_{j \in J} L_S(\beta_j) + \sum_{j \notin J} L_S(\beta_j) \right) \geq 2\pi (n - 2 + 2g),
\]
and using that \((c_{1,g}(S))^{-1} \geq 2(2J)^{-1} \sum_{j \in J} L_S(\beta_j)\) one gets
\[
\sum_{j \notin J} L_S(\beta_j) \geq \left( \frac{4\pi(n - 2 + 2g)}{2J} - 1 \right) \sum_{j \in J} L_S(\beta_j) \geq (2\pi - 1) \sum_{j \in J} L_S(\beta_j).
\]
If \( \varepsilon > 0 \) is chosen so that \( \varepsilon < \varepsilon_0 \) and using that \((4c_{1,g}(S))^{-1}\) then for any \( j \notin J \), \( \beta_j \) is in \( \partial \tilde{\Omega} \), and the above inequality implies \( L_S(\partial \tilde{\Omega}) \geq (2\pi - 1)L_S(\partial \tilde{\Omega} \setminus \partial \tilde{\Omega}) \) for any geodesic domain in \( S \).

Let us show now that \( S_\varepsilon \) (with \( \varepsilon < \min \{ \varepsilon_0, (12c_{1,g}(S))^{-1} \} \) chosen as above) satisfies the LII for intrinsic geodesic domains. If \( \Omega \) is an intrinsic geodesic domain in \( S_\varepsilon \) then it can be written as \( \tilde{\Omega} = \Omega \cap S_\varepsilon \), where \( \tilde{\Omega} \) is a geodesic domain in \( S \) such that \( \partial \tilde{\Omega} = \partial \Omega \) and since \( \Omega \) satisfies the LII in \( S \),
\[
A_S(\Omega) \leq A_S(\tilde{\Omega}) \leq c_{1,g}(S) \left( L_S(\partial \tilde{\Omega}) + L_S(\partial \tilde{\Omega} \setminus \partial \tilde{\Omega}) \right) \leq \frac{2\pi c_{1,g}(S)}{2\pi - 1} L_S(\partial \tilde{\Omega}).
\]
Then Lemma 6.5 gives the second item.

Following Kanai’s procedure, the LII will be transferred from bordered surfaces to nets and vice versa. To this end, a subset \( G \) of \( S \) is said to be \( \delta \)-separated for \( \delta > 0 \), if \( d_G(p,q) > \delta \) whenever \( p \) and \( q \) are distinct points of \( G \). It is called maximal if it is maximal with respect to the order relation of inclusion.

Consider the distance \( d_G \) in \( G \) induced by the distance \( d_S \) of \( S \). Concretely, given \( p_1, p_2 \in G \), \( d_G(p_1, p_2) = M \) if and only if \( M \geq 0 \) is the only natural number such that
\[
\delta M \leq d_G(p_1, p_2) < \delta(M + 1).
\]
The set of neighbors of \( G \) is defined as \( N(p) = \{ q \in G : d_G(p, q) = 1 \} \) and gives a net structure to the set \( G \). Such net will be referred to as \( \delta \)-net.

The linear isoperimetric inequality on nets is therefore defined as follows.

**Definition 6.7.** Let \( G \) be a net. For a subset \( T \) of \( G \), define its boundary as \( \partial T := \{ q \in G \setminus T : d_G(q, T) = 1 \} \). It is said that \( G \) satisfies the LII if there exists a finite constant \( c_1(G) \geq 0 \) so that for any non-empty finite subset \( T \) of \( G \),
\[
\#T \leq c_1(G) \#\partial T.
\]

Let \( S \) be a Riemann surface and \( 0 < \varepsilon < \arcsinh 1 \). Note that Lemma 3.2 gives that \( \iota(V_\varepsilon(S_\varepsilon)) \geq c(\varepsilon) \), where \( c(\varepsilon) := \arcsinh \left( e^{-\varepsilon} \sinh \varepsilon \right) \). The pair \((G, \delta)\) will denote a \( \delta \)-net associated to the pair \((S, \varepsilon)\) as follows: Set \( \delta \leq \frac{1}{4} \iota(V_\varepsilon(S_\varepsilon)) \), and choose a maximal \( \delta \)-net \( G \) on \( S_\varepsilon \) so that
\[
A_S(S_\varepsilon \cap B_S(p, \delta)) > \frac{1}{2} A_S(B_S(p, \delta)),
\]
for all \( p \in G \); such choice of \( G \) is possible due to the Collar Lemma. Note also that \( G \) does not need to be connected.

Notice that since \((G, \delta)\) is maximal, there are no neighborhoods of points of \( S_\varepsilon \) that are not covered by balls \( B_S(p, \delta) \) with \( p \in G \). If this were the case one could add such point \( p \) to the net \( G \) contradicting maximality. If nevertheless there was a point \( q \) on the boundary of some balls \( B_S(p, \delta) \) not covered, these balls could be slightly moved so that a neighborhood of \( q \) would not be covered and, as before, add \( q \) to the net. So, without loss of generality, \( S_\varepsilon \subset \bigcup_{p \in G} B_S(p, \delta) \).
The strategy of the proof of Theorem 1.1 is as follows: Consider $S$ and $S'$ Riemann surfaces and $f: S \to S'$ a quasi-isometry, $(G, \delta)$ and $(G', \delta')$ nets in $(S, \varepsilon)$ and $(S', \varepsilon')$. It will be assumed that $S'$ satisfies the LII that will be transferred to the net $(G', \delta')$. Then it will be shown that $(G, \delta)$ and $(G', \delta')$ are quasi-isometric and so $(G, \delta)$ also satisfies the LII. Finally, this LII will be transferred to $S$. The next two results deal with transferring the LII between surfaces and nets: A direct application of [18, Lemma 4.5] is the following result:

**Lemma 6.8.** Let $S'$ be a non-exceptional Riemann surface satisfying the LII and $0 < \varepsilon' < \min \{ \varepsilon_0, (12c_1(S'))^{-1} \}$, where $\varepsilon_0$ is the constant in Proposition 6.6. Let $(G', \delta')$ be a $\delta'$-net associated to $(S', \varepsilon')$. Then, 

$$(G', \delta') \text{ also satisfies the LII and } c_1(G') \leq \frac{12 \sinh \delta'}{\cosh(\delta'/2) - 1} c_1(S').$$

**Proof.** Proposition 6.6 implies that $S'^{\prime}$ satisfies the LII and applying Kanai's arguments in [18, Lemma 4.5] to $S'^{\prime}$ and $G'$ the proof follows. \(\Box\)

The following lemma gives the other direction. To this end, recall Buser's local lineal isoperimetric inequality ([6, p.215],[18, p.411]).

**Lemma 6.9.** Let $(G, \delta)$ be a $\delta$-net associated to $(S, \varepsilon)$. Then

$$(6.24) \quad (G, \delta) \text{ has LII } \implies S \text{ has LII}_{\varepsilon}.$$ 

Moreover, $c_1(S_{\varepsilon}) \leq 2m_{1, \ell}(S_{\varepsilon}) \max \left\{ 1, 2c_1(G) \left( \frac{\sinh(9\delta/4)}{\sinh(\delta/4)} \right)^2 \right\} + 2$, where $c_{1, \ell}(S_{\varepsilon})$ is the constant in the local LII and $m := \sup_{z \in S} \# \{ p \in G : z \in B_S(p, \delta) \} < \infty$.

**Proof.** As in the previous lemma, it is possible to reproduce Kanai's proof in [18, Lemma 4.5] to get the result, mainly because it deals with a subset of $S$ with positive injectivity radius, $S_{\varepsilon}$.

By Lemma 6.5, it suffices to consider $\Omega$ an intrinsic geodesic domain of $S_{\varepsilon}$, for which it is possible to separate $\partial S_{\varepsilon}$ from $\partial_{\ell} \Omega$ (by the Collar Lemma). That is, if $p \in \partial_{\ell} \Omega$, there exists a ball $B_S(p, 3\varepsilon)$ so that $\partial S_{\varepsilon} \cap B_S(p, 3\varepsilon) = \emptyset$. Following Kanai's proof, define sets $O, P_0 \subset G$

$$O := \left\{ p \in G : A_S(B_S(p, \delta) \cap \Omega) > \frac{1}{2} A_S(B_S(p, \delta)) \right\} \quad \text{and} \quad P_0 := \left\{ p \in G \setminus O : B_S(p, \delta) \cap \Omega \neq \emptyset \right\},$$

so that $\Omega \subseteq \bigcup_{p \in O \cup P_0} B_S(p, \delta)$.

Since $\Omega$ is an intrinsic geodesic domain its boundary is a union of simple closed curves, some of them curves of $\partial S_{\varepsilon}$ and the rest geodesics on $S$ (the latter conform $\partial_{\ell} \Omega$). Since $(G, \delta)$ is a $\delta$-net associated to $(S, \varepsilon)$, the Collar Lemma implies that if $B_S(p, \delta)$ (for $p \in G$) intersects one curve of $\partial \Omega \setminus \partial_{\ell} \Omega \subset \partial S_{\varepsilon}$ then it does not intersect any other curve of $\partial \Omega$. If this is the case, the fact that $G \subset S_{\varepsilon}$ and condition (6.23) imply that $p \in O$. Therefore, $B_S(p, \delta) \cap (\partial \Omega \setminus \partial_{\ell} \Omega) = \emptyset$ for all $p \in P_0$. Since $c(S_{\varepsilon}) > \delta$ the local LII can be applied

$$\sum_{p \in P_0} A_S(B_S(p, \delta) \cap \Omega) \leq c_{1, \ell}(S_{\varepsilon}) \sum_{p \in P_0} L_S(B_S(p, \delta) \cap \partial \Omega) = c_{1, \ell}(S_{\varepsilon}) \sum_{p \in P_0} L_S(B_S(p, \delta) \cap \partial_{\ell} \Omega) \leq c_{1, \ell}(S_{\varepsilon}) mL_S(\partial_{\ell} \Omega).$$

Now, following Kanai's estimates:

$$A_S(\Omega) \leq \sum_{p \in O} A_S(B_S(p, \delta) \cap \Omega) + \sum_{p \in P_0} A_S(B_S(p, \delta) \cap \Omega) \leq A(\delta)O + c_{1, \ell}(S_{\varepsilon}) mL_S(\partial_{\ell} \Omega),$$

where $A(r) = 4\pi \sinh^2 r$ is the area of balls with radius $r$ in $\mathbb{D}$ (the universal covering space of $S$). Writing $\nu := \frac{L_S(\partial_{\ell} \Omega)}{A_S(\Omega)}$, then $A_S(\Omega) \leq \frac{A(\delta)}{1 - c_{1, \ell}(S_{\varepsilon}) \nu}O$. If $\nu \geq (2mc_{1, \ell}(S_{\varepsilon}))^{-1}$, then the LII holds for $\Omega$ with constant $2mc_{1, \ell}(S_{\varepsilon})$; otherwise, $\nu \leq (2mc_{1, \ell}(S_{\varepsilon}))^{-1}$ and thus

$$A_S(\Omega) \leq 2A(\delta)O.$$
On the other hand, points in $\partial_2 \Omega$ will be near of points in $\partial \Omega$ (since $\partial \Omega \subset S_\ast$). More precisely, if $p \in \partial \Omega$ then there exist $p' \in N(p) \cap \partial \Omega$, and $B(p, \delta) \cap \sigma \neq \emptyset$ for some simple closed geodesic $\sigma \subset \partial_1 \Omega$. Note that $\sigma$ depends on $p$. Thus $d_\delta(p', \sigma) < 2\delta$ and therefore $\partial_2 \Omega \subset V_{2\delta}(\partial_1 \Omega)$. Let $Q$ be a maximal $\delta$-separated subset of $\partial_2 \Omega$; then $\cup_{p \in Q} B_\delta(p, \delta/2) \subset V_{\delta/2}(\partial_1 \Omega) \subset \cup_{q \in Q} B_\delta(q, \delta/2)$, which implies

$$A(\delta/2)\partial \Omega \leq \sum_{q \in Q} A_\delta (B_\delta(q, 9\delta/2)) \leq \frac{A(9\delta/2)}{A(\delta)} \sum_{q \in Q} A_\delta (B_\delta(q, \delta)) = \frac{2A(9\delta/2)}{A(\delta)} \sum_{q \in Q} A_\delta (B_\delta(q, \delta) \cap \Omega),$$

where the local isoperimetric inequality was once again used. Combining this estimate with the previous one, and using the LII for $G$ the desired result is obtained also in the case $\nu \leq (2mc_{1,1}(S_\ast))^{-1}$. \hfill \Box

As a last step, it will be constructed a quasi-isometry between the two nets $(G, \delta)$ and $(G', \delta')$ associated to $(S, \varepsilon)$ and $(S, \varepsilon')$ respectively with $0 < \varepsilon < \arcsinh 1$ and $0 < \varepsilon', \tilde{\varepsilon} < \varepsilon$ given by Lemma 6.1.

**Proposition 6.10.** The nets $(G, \delta)$ and $(G', \delta')$ are quasi-isometric. More precisely, there is a $C'$-full $(A, B)$-quasi-isometry $g : G \rightarrow G'$, with $A = \max \left\{ \frac{\delta'}{\delta}, \frac{\delta}{\delta'} \right\}$, $B = 5 + \frac{a\delta}{\delta'} + \frac{b}{\delta'}$ and $C' = 2 + \frac{a(2\delta + C(\varepsilon, \tilde{\varepsilon})) + 2b + c}{\delta'}$ where $C(\varepsilon, \tilde{\varepsilon})$ is the maximum diameter of the connected components of $S_\ast \setminus S_{\varepsilon'}$ where $\tilde{\varepsilon}$ is given by Lemma 6.1.

Moreover, for any $X \subset G$, $\#X \leq \mu \#g(X)$ where $\mu \leq 13(\frac{a(2\delta + A) + 1}{\delta + 1}).$

**Remark 6.11.** No connectivity is assumed for either $G$ or $G'$. Note that the constant $C(\varepsilon, \tilde{\varepsilon})$ does not depend on $S$ due to Margulis Lemma.

In [18, Lemma 4.2] Kanai proves that the LII on graphs is preserved by quasi-isometries; thus an immediate consequence is:

**Corollary 6.12.** For $(G, \delta)$ and $(G', \delta')$ as above,

$$(G, \delta) \text{ satisfies the LII } \iff (G', \delta') \text{ satisfies the LII.}$$

Moreover, $c_1(G) \leq \mu 12^A(B + 2C - 1) + C - 2c_1(G')$, with $\mu$ as in Proposition 6.10.

**Proof.** The function $g$ will be defined as follows:

Given $p_1 \in G$, there exists at least one point $p'_1 \in G'$ so that $p'_1 \in B_\delta(f(p_1), 2\delta')$, since $f(S_\ast) \subset S_{\varepsilon'}$ by Lemma 6.1. Define $g(p_1) := p'_1$.

Consider two points $p_1, p_2 \in G$ and suppose $d_{G'}(g(p_1), g(p_2)) = M$ for some $M \geq 0$; that is,

$$M\delta' \leq d_{S'}(g(p_1), g(p_2)) < (M + 1)\delta'.$$

Transferring this property to $f$:

$$d_{S'}(f(p_1), f(p_2)) \leq d_{S'}(f(p_1), g(p_1)) + d_{S'}(f(p_2), g(p_2)) + d_{S'}(g(p_1), g(p_2)) \leq 4\delta' + (M + 1)\delta' = (5 + M)\delta'.$$

This estimate together with $f$ being an $(a, b)$-quasi-isometry give:

$$\frac{1}{a} d_{S'}(p_1, p_2) - b \leq d_{S'}(f(p_1), f(p_2)) \leq (5 + M)\delta' \leq \delta'(5 + d_{G'}(g(p_1), g(p_2))),$$

that is,

$$\frac{1}{a}\delta' d_{S'}(p_1, p_2) - \left( \frac{b}{\delta'} + 5 \right) \leq d_{G'}(g(p_1), g(p_2)).$$
Let Theorem 2.2 in [32] gives the first statement. S

In order to prove the second one, assume that S does not satisfy the LII. Hence, by Lemma 6.5 there exists a sequence of geodesic domains \( \Omega_n \) in \( S \) with \( A_S(\Omega_n)/L_S(\partial \Omega_n) \to \infty \). Since \( S_0 \) satisfies the LII, without loss of generality, assume that there exists \( 1 \leq j_n \leq k \) with \( \sigma_{j_n} \subset \partial \Omega_n \) for each \( n \); furthermore, since
$L_S(\sigma_1 \cup \cdots \cup \sigma_k)$ is a fixed number (and then bounded), one can also assume $A_S(\Omega_n) < c$ for some constant $c$ and every $n$, and $L_S(\partial \Omega_n) \to 0$ (note that $L_S(\partial \Omega_n) > 0$ since $S$ has infinite area).

Let us consider a ball $B_S(z, r)$ in $S$ with $\sigma_1 \cup \cdots \cup \sigma_k \subset B_S(z, r)$; let us choose now $R > r$ with $A_S(B_S(z, R) \setminus B_S(z, r)) > c$ (this is possible since $S$ has infinite area). Let us define $u := \min \{ L_S(\sigma) : \sigma$ is a simple closed geodesic with $\sigma \cap B_S(z, R) \neq \emptyset \}$. Since $A_S(B_S(z, R) \setminus B_S(z, r)) > A_S(\Omega_n)$ and $\sigma_1 \cup \cdots \cup \sigma_k \subset B_S(z, r)$, there exists a simple closed geodesic $\sigma^n \subset \partial \Omega_n$ with $\sigma^n \cap B_S(z, R) \neq \emptyset$ and $\sigma^n \neq \sigma_j$ for $j = 1, \ldots, k$ (since $S \setminus \{ \sigma_1 \cup \cdots \cup \sigma_k \}$ is connected and $S$ has infinite area, there is no geodesic domain $\Omega$ in $S$ with $\partial \Omega \subseteq \sigma_1 \cup \cdots \cup \sigma_k$). Hence,

$$A_S(\Omega_n) \leq c \frac{L_S(\sigma^n)}{u} \leq \frac{c}{u} L_S(\partial \Omega_n),$$

which contradicts $A_S(\Omega_n)/L_S(\partial \Omega_n) \to \infty$. \hfill $\square$

**Theorem 7.2.** Let $S$ and $S'$ be quasi-isometric non-exceptional Riemann surfaces with finite genus. Then $S'$ satisfies the LII if and only if $S$ satisfies the LII.

**Proof.** It is not difficult to check that $S$ has finite area if and only if $S'$ has finite area; in this case, $S$ and $S'$ do not satisfy the LII. Otherwise, the theorem is a consequence of Lemma 7.1 and Theorem 1.1 (which also holds for bordered surfaces whose border is a finite union of simple closed geodesics; it suffices to take $\varepsilon, \varepsilon', \xi$ less than half the minimum of the lengths of these simple closed geodesics).

It is not possible to obtain a quantitative version of Theorem 7.2, as shows the following example.

**Example 7.3.** There exist constants $a, b, c$, with the following property: for each $n$ there exist non-exceptional Riemann surfaces with finite genus $S_n$ satisfying the LII and a $c$-full $(a, b)$-quasi-isometry $f_n : S_n \to S_1$, with $\lim_{n \to \infty} c_1(S_n) = \infty$.

Let us consider two isometric $Y$-pieces $Y_1, Y_2$ such that $\partial Y_j$ is the union of three simple closed geodesics with length 1 for $j = 1, 2$. Denote by $X$ the bordered surface obtained by pasting two boundary curves of $Y_1$ with two boundary curves of $Y_2$ ($X$ is a torus with two holes). Let us consider a sequence $(X_m)_{m \geq 1}$ of bordered surfaces isometric to $X$; denote by $R_n$ the bordered surface obtained from $X_1, \ldots, X_n$ by pasting a boundary curve of $X_m$ with a boundary curve of $X_{m+1}$ for every $1 \leq m \leq n-1$ ($R_n$ is a surface with genus $n$ and two boundary curves). Consider now a generalized $Y$-piece $Y_0$ with a cusp and such that $\partial Y_0$ is the union of two simple closed geodesics with length 1. Denote by $R_0$ the bordered surface obtained by pasting two boundary curves of $Y_1$ with two boundary curves of $Y_0$ ($R_0$ is a torus with a cusp and a hole). $S_n$ is the (non bordered) surface obtained by pasting a funnel (with boundary of length 1) to one boundary curve of $R_n$ and $R_0$ to the other boundary curve of $R_n$.

$S_n$ satisfies the LII since a surface of finite type satisfies the LII if and only if it has at least a funnel.

The domain $\bigcup_{m=1}^n X_m$ in $S_n$ has area $4\pi n$ and its boundary has length 2 for every $n \geq 1$. This implies that $\lim_{n \to \infty} c_1(S_n) = \infty$.

8. Non-linear isoperimetric inequalities

This section deals with $\alpha$-isoperimetric inequalities with $1/2 \leq \alpha < 1$, which have a very different behavior from LII.

**Proposition 8.1.** If a Riemann surface $S$ satisfies $t(S) = 0$, then $S$ does not satisfy the $\alpha$-isoperimetric inequality for each $1/2 \leq \alpha < 1$.

**Proof.** Seeking for a contradiction, let us assume that $S$ satisfies the $\alpha$-isoperimetric inequality for some $1/2 \leq \alpha < 1$.

If $S$ has a cusp, let us consider the $a$-collars $C(a)$ of the cusp, with $0 < a \leq 2$. It is well known that $A_S(C(a)) = L_S(\partial C(a)) = a$; hence, $\alpha a^\alpha \leq c_{\alpha} a$, which gives a contradiction if $a \to 0^+$.\hfill $\square$

If $S$ has no cusp, then there exists a sequence of simple closed geodesics $(\sigma_n)$ with $\lim_{n \to \infty} L_S(\sigma_n) = 0$. Denote by $C_n$ the collar of $\sigma_n$ of width 1. It is well known that $A_S(C_n) = 2L_S(\sigma_n)\sinh 1$ and $L_S(\partial C_n) = 2L_S(\sigma_n)\cosh 1$; hence, $(2L_S(\sigma_n)\sinh 1)^\alpha \leq c_{\alpha} 2L_S(\sigma_n)\cosh 1$, which gives a contradiction if $n \to \infty$.\hfill $\square$
From Proposition 8.1 and [18, Theorem 4.1] the following result is deduced.

**Theorem 8.2.** Let $S$ and $S'$ be quasi-isometric non-exceptional Riemann surfaces with $\iota(S) > 0$, and $1/2 \leq \alpha < 1$. Then $S'$ satisfies the $\alpha$-isoperimetric inequality if and only if $S$ satisfies the $\alpha$-isoperimetric inequality and $\iota(S') > 0$.

**References**


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