HYPERBOLICITY AND COMPLEMENT OF GRAPHS

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Abstract. If $X$ is a geodesic metric space and $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1 x_2], [x_2 x_3]$ and $[x_3 x_1]$ in $X$. The space $X$ is $\delta$-hyperbolic (in the Gromov sense) if for any side of $T$ is contained in a $\delta$-neighborhood of the union of the two other sides, for every geodesic triangle $T$ in $X$. We denote by $\delta(X)$ the sharp hyperbolicity constant of $X$, i.e. $\delta(X) := \inf(\delta \geq 0 : X$ is $\delta$-hyperbolic). The study of hyperbolic spaces is interesting since the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it. The main aim of this paper is to obtain information about the hyperbolicity constant of the complement graph $G$ in terms of properties of the graph $G$. In particular, we prove that if $\text{diam} V(G) \geq 3$, then $\delta(T) \leq 2$, and that the inequality is sharp. Furthermore, we find several relations between $\delta(G)$ and $\delta(T)$. For instance, if the girth of $G$ satisfies $g(G) \geq 8$, then $\delta(T) \leq \delta(G)$. Besides, we bound $\delta(G) + \delta(T)$ and $\delta(G) \delta(T)$.

Keywords: Graph; Complement; Connectivity; Geodesic; Gromov hyperbolicity.

AMS Subject Classification numbers: 05C69; 05A20; 05C50

1. Introduction.

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [1, 2, 5, 6, 12, 13, 14, 15, 16, 17, 19, 21, 22, 23]. The theory of Gromov’s spaces was used initially for the study of finitely generated groups, where it was demonstrated to have an enormous practical importance. This theory was applied principally to the study of automatic groups (see [18]), that play an important role in sciences of the computation. Another important application of this spaces is secure transmission of information by internet (see [12, 13, 14]). In particular, the hyperbolicity also plays an important role in the spread of viruses through the network (see [13, 14]). The hyperbolicity is also useful in the study of DNA data (see [5]).

The study of Gromov hyperbolicity in Riemann surfaces with their Poincaré metrics is the subject of [9, 10, 11, 19, 20, 22, 23]. In particular, in [19, 22, 23] it is proved the equivalence of the hyperbolicity of Riemann surfaces and the hyperbolicity of a simple graph; hence, it is useful to know hyperbolicity criteria for graphs.

In our study on hyperbolic graphs we use the notations of [8]. We give now the basic facts about Gromov’s spaces. If $\gamma : [a, b] \longrightarrow X$ is a continuous curve in a metric space $(X, d)$, we say that $\gamma$ is a geodesic if it is an isometry, i.e. $L(\gamma|_{[s, t]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$. We say that $X$ is a geodesic metric space if for every $x, y \in X$ there exists a geodesic joining $x$ and $y$; we denote by $[xy]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If $X$ is a graph, we use the classical notation $uv$ for the edge of a graph joining the vertices $u$ and $v$.

In order to consider a graph $G$ as a geodesic metric space, we must identify any edge $uv \in E(G)$ with the real interval $[0, l]$ (if $l := L(uv)$); hence, if we consider $uv$ as a graph with just one edge, then it is isometric to $[0, l]$. Therefore, any point in the interior of the edge $uv$ is a point of $G$. A connected graph $G$ is naturally equipped with a distance defined on its points, induced by taking shortest paths in $G$. Then, we see $G$ as a metric graph. Along the paper we just consider simple graphs whose edges have length 1. These conditions guarantee that each connected component of the graph is a geodesic space.

Date: August 2, 2010.

1
If $X$ is a geodesic metric space and $J = \{J_1,J_2,\ldots,J_n\}$ is a polygon, with sides $J_i \subseteq X$, we say that $J$ is $\delta$-thin if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_i) \leq \delta$. We denote by $\delta(J)$ the sharp thin constant of $J$, i.e. $\delta(J) := \inf\{\delta \geq 0 : J \text{ is } \delta\text{-thin}\}$. If $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1,x_2,x_3\}$ is the union of the three geodesics $[x_1,x_2], [x_2,x_3]$ and $[x_3,x_1]$. The space $X$ is $\delta$-hyperbolic (or satisfies the Rips condition with constant $\delta$) if every geodesic triangle in $X$ is $\delta$-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of $X$, i.e. $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. We say that $X$ is hyperbolic if $X$ is $\delta$-hyperbolic for some $\delta \geq 0$. If $X$ is hyperbolic, then $\delta(X) = \inf\{\delta \geq 0 : X \text{ is } \delta\text{-hyperbolic}\}$. If $X$ is not connected but their connected components $\{X_n\}$ are geodesic spaces, then we define $\delta(X) := \sup_n \delta(X_n)$.

2. Hyperbolicity constants of a graph and its complement.

Given any simple finite or infinite graph $G$ with edges of length 1, we denote by $\overline{G}$ the complement of $G$.

The main aim of this paper is to obtain information about the hyperbolicity constant of $G$ (respectively, $\overline{G}$) in terms of properties of $G$ (respectively, $\overline{G}$).

In [21, Theorem 8] we find the following result.

**Lemma 2.1.** In any graph $G$ the inequality $\delta(G) \leq \frac{1}{2} \text{ diam } G$ holds, and furthermore, it is sharp.

We start with some examples. The following graphs have these precise values of $\delta$:

- The complement of the path graphs verify $\delta(\overline{P_n}) = 5/4$ for every $n \geq 5$.
- The complement of the cycle graphs verify $\delta(\overline{C_n}) = 5/4$ for every $n \geq 5$.
- The complement of the star graphs verify $\delta(\overline{S_n}) = 1$ for every $n \geq 5$.

Furthermore, these graphs verify $\delta(G) = 1/2 \text{ diam } G$.

**Theorem 2.2.** If $\text{ diam } V(G) \geq 3$, then $\overline{G}$ is connected and $\delta(\overline{G}) \leq 2$, and this inequality is sharp.

**Proof.** It is well known that if $\text{ diam } V(G) \geq 3$, then $\overline{G}$ is connected and $\text{ diam } V(\overline{G}) \leq 3$. Then $\text{ diam } \overline{G} \leq \text{ diam } V(G) + 1 \leq 4$, and Lemma 2.1 gives the inequality. The following graph shows that this inequality is sharp: $\overline{G}$ is a graph with ten vertices whose edges are: $(1,2),(1,5),(1,10),(2,3),(2,9),(2,10),(3,4),(3,9),(3,10), (4,8),(4,9),(5,6),(5,10),(6,7),(6,9),(6,10),(7,8),(7,9),(7,10),(8,9),(9,10)$.

It is not possible to obtain a reciprocal of Theorem 2.2, as the following example shows: if $n \geq 5$ and $G = P_n$, then $\delta(\overline{G}) = 0$ and $\text{ diam } V(G) = 2$.

Using an argument similar to the one in the proof of Theorem 2.2, we obtain the following result, which relates connectivity and hyperbolicity.

**Theorem 2.3.** If $\delta(G) > 3/2$, then $\overline{G}$ is connected.

The following result is well known.

**Lemma 2.4.** If $m \geq 2$ is a natural number and $G$ is any finite graph with $\text{ deg } v \geq m$ for every $v \in V(G)$, then there exists a simple cycle $\eta$ in $G$ with $L(\eta) \geq m + 1$.

From [21, Proposition 5 and Theorem 7] we deduce the following results.

**Lemma 2.5.** If $G$ is any graph with a cycle $g$ with length $L(g) \geq 3$, then $\delta(G) \geq 3/4$. If there exists a cycle $g$ in $G$ with length $L(g) \geq 4$, then $\delta(G) \geq 1$.

Lemmas 2.4 and Lemma 2.5 give directly the following proposition.

**Proposition 2.6.** Let $G$ be any graph with $n$ vertices.

- If the maximum degree of $G$ satisfies $\Delta \leq n - 3$, then $\delta(\overline{G}) \geq 3/4$.
- If the maximum degree of $G$ satisfies $\Delta \leq n - 4$, then $\delta(\overline{G}) \geq 1$.

Both inequalities are sharp: the first inequality is attained for the graph with 3 vertices without edges, and the second one for the graph with $n \geq 4$ vertices without edges.

The following Theorem gives precise bounds of $\delta(G)$ for graphs with large minimum degree.
Theorem 2.7. If $G$ is any graph with $n$ vertices and $\deg v \geq (n-1)/2$ for every $v \in V(G)$, then $\delta(G) \leq 3/2$.
Furthermore, $\delta(G) \geq 1$ if $n \geq 6$ and $\delta(G) \geq 3/4$ if $n = 5$.

Proof. Let us fix an arbitrary vertex $v \in V(G)$ and let us define $N(v) := \{u \in V(G) : d(u, v) = 1\}$. By hypothesis, $N(v)$ has at least $(n-1)/2$ elements. Let us consider any vertex $w \in V(G) \setminus (N(v) \cup \{v\})$. Since $\deg w \geq (n-1)/2$, $v, w \notin N(w)$ and $(n-1)/2 + (n-1)/2 + 2 \geq n + 1$, we deduce that $N(v) \cap N(w) \neq \emptyset$. Consequently, $d(w, v) = 2$ and therefore we conclude that $\text{diam}(G) = 2$. Hence, $\text{diam}(G) \leq 3$, and then $\delta(G) \leq 3/2$ by Lemma 2.1.

Furthermore, Proposition 2.6 gives $\delta(G) \geq 1$ if $n \geq 6$, and $\delta(G) \geq 3/4$ if $n = 5$. □

Note that it is not possible to improve the lower bound in Theorem 2.7: the complete graphs $K_n$ verify $\delta(K_n) = 1$ for every $n \geq 6$ (see [21, Theorem 11]), and the graph $G$ with five vertices obtained by pasting two graphs isomorphic to $C_3$ (identifying a vertex of one copy of $C_3$ with a vertex of the other copy of $C_3$) verifies $\delta(G) = 3/4$. The upper bound is sharp, as shows the graph $G$ with eight vertices whose edges are: (1,2),(1,6),(1,7),(1,8),(2,3),(2,7),(2,8),(3,4),(3,7),(3,8),(4,5),(4,7),(4,8),(5,6),(5,7),(5,8),(6,7),(6,8),(7,8).

We have the following direct consequence of Theorem 2.7.

Corollary 2.8. If $G$ is any graph with $n$ vertices and maximum degree $\Delta \leq (n-1)/2$, then $\delta(G) \leq 3/2$.
Furthermore, $\delta(G) \geq 1$ if $n \geq 6$ and $\delta(G) \geq 3/4$ if $n = 5$.

Theorem 2.9. If $G$ is any graph containing a set $U \subseteq V(G)$ with five vertices such that the subgraph of $G$ spanned by $U$ is a forest, then $\delta(G) \geq 1$.

Proof. First of all, we will prove that there exists a simple cycle $\eta$ in $G$ with every vertex of $\eta$ in $U$ and $L(\eta) \geq 4$. Without loss of generality we can assume that the subgraph of $G$ spanned by $U$ is a tree, since otherwise we can add some edges to the forest in order to obtain a tree. Now, it suffices to check that the complements of the three different trees (up to graph isomorphism) with five vertices contain a simple cycle $\eta$ joining vertices in $U$ with $L(\eta) \geq 4$ (since $\eta$ is contained in $G$).

Since there exists a simple cycle $\eta$ in $G$ with $L(\eta) \geq 4$, Corollary 2.10 gives that $\delta(G) \geq 1$. □

We obtain directly the following corollary.

Corollary 2.10. If $G$ is any forest with at least five vertices, then $\delta(G) \geq 1$.

Taking as $G$ the star graph with at least five vertices, we check that the inequality in Corollary 2.10 is sharp. Hence, the inequality in Theorem 2.9 is also sharp.

Theorem 2.11. Let $G$ be any tree. If either $G$ has at most three vertices or $G$ is isomorphic to $P_4$, then $\delta(G) = 0$; if $G$ is isomorphic to the star with four vertices, then $\delta(G) = 3/4$; otherwise, $1 \leq \delta(G) \leq 2$.
Furthermore, $G$ is connected if $G$ is not a star graph.

Proof. If either $G$ has at most three vertices or $G$ is isomorphic to $P_4$, then $G$ is a forest and $\delta(G) = 0$.

If $G$ is a tree with $\text{diam}(G) = 2$, then $G$ is isomorphic to a star graph $S_n$ (with a central vertex); in this case, $G$ has two connected components, one of them a single vertex (with $\delta = 0$), and the other isomorphic to $K_{n-1}$ (with $\delta = 3/4$ if $n = 4$, and $\delta = 1$ if $n > 4$). Therefore, $\delta(G) = 3/4$ if $n = 4$, and $\delta(G) = 1$ if $n > 4$.

If $G$ is not isomorphic to $P_4$ and $\text{diam}(G) \geq 3$, then $G$ has at least five vertices and, Corollary 2.10 gives $\delta(G) \geq 1$. Theorem 2.2 gives the inequality $\delta(G) \leq 2$. □

Theorem 2.3 and Corollary 2.10 give the following theorem for forests.

Theorem 2.12. If $G$ is any non-connected forest with at least five vertices, then $1 \leq \delta(G) \leq 3/2$.

Theorem 2.13. If $\delta(G) > 2$, then $G$ is $k$-connected with $k \geq 2$. 
Assume that $G$ is either non-connected or 1-connected.

If $G$ is non-connected, then Theorem 2.3 gives that $\delta(G) \leq 3/2$.

Assume now that $G$ is 1-connected and let $v$ be a vertex with $G \setminus \{v\}$ not connected. Let us denote by $V_1, V_2, \ldots, V_k$ the connected components of $V(G)$ obtained by removing $v$.

Assume first that $vw \in E(G)$ for every $w \in V(G) \setminus \{v\}$. In this case, $G$ has two connected components, and $\{v\}$ is one of them; we denote by $G_0$ the other connected component. It is clear that if $v_i \in V_i$ and $v_j \in V_j$, with $i \neq j$, then $v_i v_j \in E(G)$; hence, $\text{diam } V(G_0) \leq 2$, $\text{diam } G_0 \leq 3$ and $\delta(G_0) \leq 3/2$ by Lemma 2.1. Since $\delta(\{v\}) = 0$, we have $\delta(G) \leq 3/2$.

Assume now that there exists $w \in V(G) \setminus \{v\}$ with $vw \notin E(G)$. In this case, $G$ is connected, since $vw \in E(G)$, and the previous argument gives now $\text{diam } V(G) \leq 3$, $\text{diam } G \leq 4$ and $\delta(G) \leq 2$. \hfill $\square$

3. Comparison of hyperbolicity constants.

**Theorem 3.1.** If $G$ is any graph with $\delta(G) \geq 2$, then $G$ is connected and $\delta(G) \leq 2$. Furthermore, if $\delta(G) > 2$, then $1 \leq \delta(G) \leq 2$. In particular, if $\delta(G) \geq 2$, then $\delta(G) \leq \delta(G)$; if $\delta(G) > 2$, then $\delta(G) < \delta(G)$.

**Proof.** By Lemma 2.1, we know that $2 \leq \delta(G) \leq 1/2 \text{ diam } G \leq 1/2(\text{diam } V(G)+1)$, and we conclude $\text{diam } V(G) \geq 3$. By Theorem 2.2 we have that $G$ is connected and $\delta(G) \leq 2$.

Furthermore, if $\delta(G) > 2$, then the previous argument gives $\text{diam } V(G) > 3$ and $\text{diam } V(G) \geq 4$. Since $\text{diam } V(G) \geq 4$, then $G$ contains a cycle $C$ with length $L(C) \geq 4$. By Lemma 2.5 we have that $\delta(G) \geq 4$. \hfill $\square$

As a consequence of Theorem 3.1, we obtain the following corollaries.

**Corollary 3.2.** If $G$ contains an isometric cycle $C$ with $L(C) \geq 8$, then $\delta(G) \leq \delta(G)$. Furthermore, if $L(C) \geq 9$, then $\delta(G) < \delta(G)$.

**Corollary 3.3.** If the girth of $G$ satisfies $g(G) \geq 8$, then $\delta(G) \leq \delta(G)$; $\delta(G) \leq \delta(G)$, $\delta(G) < \delta(G)$.

In the proof of Theorem 3.5 below we will need the following result (see [16, Proposition 27]).

**Lemma 3.4.** Let $G$ be any graph with $n \geq 4$ vertices. If $\text{deg } v \geq n-2$ for every vertex $v \in V(G)$, then $\delta(G) = 1$ and $\text{diam } G = 2$.

**Theorem 3.5.** If $G$ is any graph with $n$ vertices and $\text{deg } v \geq n-2$ for every vertex $v \in V(G)$, then $\delta(G) < \delta(G)$.

**Proof.** By Lemma 3.4, we have $\delta(G) = 1$. We have that $G$ is a forest, and then $\delta(G) = 0$. \hfill $\square$

Note that if $\text{deg } v \geq n-3$ for every vertex $v \in V(G)$, the conclusion of Theorem 3.5 does not hold: it suffices to consider the graph $G$ with $G = C_n$.

**Theorem 3.6.** If $G$ is any graph with $\delta(G) > \delta(G) + 1/2$, then $G$ is connected or $\delta(G) \leq 3/4$.

**Proof.** Assume that $G$ is not connected and $\delta(G) > 3/4$.

Since $G$ is not connected, Theorem 2.3 gives $\delta(G) \leq 3/2$.

Since $\delta(G) > 3/4$, [16, Theorem 11] gives that there exists a cycle $g$ in $G$ with length $L(g) \geq 4$; then Lemma 2.5 gives that $\delta(G) \geq 1$.

Hence, $\delta(G) \leq 3/2 \leq \delta(G) + 1/2$. \hfill $\square$

Notice that if $G$ is any graph with five vertices and $\delta(G) = 1$, then $\delta(G) < \delta(G)$.

**Theorem 3.7.** Let $G$ be any graph with $n \geq 5$ vertices. If for any cycle $C$ contained in $G$, $L(C) \leq 3$, then $\delta(G) \leq \delta(G)$. 

Proof. If $G$ is a tree, then $\delta(G) = 0 \leq \delta(\overline{G})$. Otherwise, $\delta(G) = 3/4$ by [16, Theorem 11]. Since $n \geq 5$, then at least one of the following cases occurs.

Case (a): $G$ contains two edges at a distance greater than or equal to two; therefore $\overline{G}$ contains a cycle of order 4 and $\delta(\overline{G}) \geq 1$ by Lemma 2.5.

Case (b): $G$ contains three independent vertices; therefore $\overline{G}$ contains a cycle of order 3 and $\delta(\overline{G}) \geq 3/4$ by Lemma 2.5. Thus, in both cases $\delta(\overline{G}) \geq 3/4$ and this completes the proof. \(\square\)

Theorem 30 in [16] gives the following result; it will be useful in the proofs of Theorems 3.10 and 3.9 below.

**Theorem 3.8.** If $G$ is any graph with $n$ vertices, then $\delta(G) \leq n/4$.

**Theorem 3.9.** If $G$ is any graph with $n$ vertices, then

$$
\delta(G) \delta(\overline{G}) \leq \begin{cases} 
0, & \text{if } n \leq 4, \\
n^2/16, & \text{if } n = 5, 6, 7, \\
n/2, & \text{if } n \geq 8.
\end{cases}
$$

Proof. If $n \leq 4$, then $G$ or $\overline{G}$ is a forest and therefore $\delta(G) \delta(\overline{G}) = 0$.

Theorem 3.8 gives that $\delta(G) \leq n/4$ for every graph $G$ with $n$ vertices; hence, we also have $\delta(\overline{G}) \leq n/4$ and $\delta(G) \delta(\overline{G}) \leq n^2/16$. This gives the inequality for $n = 5, 6, 7$.

Consider now the case $n \geq 8$. By Theorem 3.1, if $G$ is a graph with $\delta(G) \geq 2$, then $\delta(\overline{G}) \leq 2$. Theorem 3.8 gives that $\delta(G) \leq n/4$, and we conclude $\delta(G) \delta(\overline{G}) \leq n/2$. Similarly, if $\delta(\overline{G}) \geq 2$, then $\delta(G) \delta(\overline{G}) \leq n/2$. If $\delta(G) \leq 2$ and $\delta(\overline{G}) \leq 2$, then $\delta(G) \delta(\overline{G}) \leq 4 \leq n/2$ and we have the result. \(\square\)

Note that we can not improve the trivial lower bound $\delta(G) \delta(\overline{G}) \geq 0$, since it is attained for any tree.

**Theorem 3.10.** If $G$ is any graph with $n$ vertices, then

$$
\delta(G) + \delta(\overline{G}) \leq \begin{cases} 
0, & \text{if } n \leq 2, \\
n/4, & \text{if } n = 3, 4, \\
n/2, & \text{if } n = 5, 6, 7, \\
2 + n/4, & \text{if } n \geq 8.
\end{cases}
$$

Besides, $\delta(G) + \delta(\overline{G}) \geq 1$ for every $n \geq 5$.

Proof. Theorem 3.8 gives that $\delta(G) \leq n/4$ and $\delta(\overline{G}) \leq n/4$; hence, $\delta(G) + \delta(\overline{G}) \leq n/2$. This gives the upper bound for $n = 5, 6, 7$.

If $n \geq 8$, the argument in the proof of Theorem 3.9 gives that $\delta(G) \delta(\overline{G}) \leq 2 + n/4$.

If $n \leq 2$, then $\delta(G) = \delta(\overline{G}) = 0$.

If $n = 3$, then we have either $\delta(G) = 0$ or $\delta(G) = 3/4$. If $\delta(G) = 0$, then Theorem 3.8 gives $\delta(\overline{G}) \leq 3/4$; if $\delta(G) = 3/4$, then $G$ is isometric to $C_3$ and $\delta(\overline{G}) = 0$; in any case we have $\delta(G) + \delta(\overline{G}) \leq 3/4 = n/4$.

If $n = 4$, then we have either $\delta(G) = 1$ or $\delta(G) \leq 3/4$ by [16, Theorem 11]. If $\delta(G) = 1$, then $G$ contains a subgraph isometric to $C_4$ and $\delta(\overline{G}) = 0$; in a similar way, if $\delta(\overline{G}) = 1$, then $\delta(G) = 0$. Otherwise, $\delta(G), \delta(\overline{G}) \leq 3/4$; if $\delta(G) = 3/4$, then $G$ contains a subgraph isometric to $C_3$ and $\delta(\overline{G}) = 0$; in a similar way, if $\delta(\overline{G}) = 3/4$, then $\delta(G) = 0$. Therefore, in any case we have $\delta(G) + \delta(\overline{G}) \leq 1 = n/4$.

Assume now that $n \geq 5$. If $\delta(G) = 0$, then $G$ is a tree with at least five vertices, and Corollary 2.10 gives that $\delta(\overline{G}) \geq 1$. Similarly, if $\delta(\overline{G}) = 0$, then $\delta(G) \geq 1$. If $\delta(G), \delta(\overline{G}) > 0$, then [16, Theorem 11] gives that $\delta(G), \delta(\overline{G}) \geq 3/4$. Hence, in any case we have $\delta(G) + \delta(\overline{G}) \geq 1$. \(\square\)

Note that the equality $\delta(G) + \delta(\overline{G}) = n/2$ holds if we have either $n = 5$ or $n = 6$; if $G = C_5$, then $\overline{G}$ is isomorphic to $G$ and thus $\delta(G) + \delta(\overline{G}) = 5/4 + 5/4 = 5/2 = n/2$; if $G = P_2 \times P_3$, then $\delta(G) + \delta(\overline{G}) = 2 + n/4 = n/2$.\(\square\)
$3/2 + 3/2 = 6/2 = n/2$. The upper bound is also attained if $n \leq 4$, as we have seen in the proof of Theorem 3.10. The lower bound is also sharp: it is attained by the complete graph $G = K_n$ with $n \geq 5$. If $n \leq 4$, then we can not improve the trivial inequality $\delta(G) + \delta\left(\overline{G}\right) \geq 0$, since it is attained with $G = P_n$.

References