Abstract. If $X$ is a geodesic metric space and $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in $X$. The space $X$ is $\delta$-hyperbolic (in the Gromov sense) if any side of $T$ is contained in a $\delta$-neighborhood of the union of the other two sides, for every geodesic triangle $T$ in $X$. The study of hyperbolic graphs is an interesting topic since the hyperbolicity of a geodesic metric space is equivalent to the hyperbolicity of a graph related to it. One of the main problems on this subject is to relate the hyperbolicity with other properties on graph theory. In this paper we extend in two ways (edge-chordality and path-chordality) the classical definition of chordal graphs in order to relate this property with Gromov hyperbolicity. In fact, we prove that every edge-chordal graph is hyperbolic and that every hyperbolic graph is path-chordal. Furthermore, we prove that every path-chordal cubic graph (with small path-chordality constant) is hyperbolic.

Key words and phrases: Chordal graph, Gromov hyperbolicity, Gromov hyperbolic graph, Infinite Graphs, Geodesics.

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1. Introduction

Hyperbolic spaces play an important role in geometric group theory and in geometry of negatively curved spaces. The concept of Gromov hyperbolicity grasps the essence of both negatively curved spaces like the classical hyperbolic space or Riemannian manifolds of negative sectional curvature, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1, 21, 22]).

The theory of Gromov spaces was used initially for the study of finitely generated groups (see [22] and the references therein), where its practical importance was discussed. This theory was mainly applied to the study of automatic groups (see [35]), which appear in computational science. The concept of hyperbolicity appears also in discrete mathematics, in particular, a few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [16, 18, 20, 32]). Another important application of these spaces is secure transmission of information on the internet (see [26, 27, 28]). In particular, the hyperbolicity plays an important role in the spread of viruses through the network (see [27, 28]). The hyperbolicity is also useful in the study of DNA data (see [8]). It has been shown empirically in [47] that the internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension.

The study of mathematical properties of Gromov hyperbolic spaces and its applications is a topic of recent and increasing interest in graph theory; see, for instance [3, 4, 5, 8, 10, 12, 19, 26, 27, 28, 29, 30, 31, 33, 34, 37, 38, 39, 40, 44, 45, 46, 48, 49].
In recent years several researchers have been interested in showing that metrics used in geometric function theory are Gromov hyperbolic. For instance, the Gehring-Osgood \( j \)-metric is Gromov hyperbolic; and the Vuorinen \( j \)-metric is not Gromov hyperbolic except in the punctured space (see [23]). The study of Gromov hyperbolicity of the quasi-hyperbolic and the Poincaré metrics is the subject of [2, 6, 24, 25, 40, 41, 42, 45, 46]. In particular, in [40, 45, 46, 48] it is proved the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a very simple graph; hence, it is useful to know hyperbolicity criteria for graphs.

We say that \( \gamma \) is a \emph{geodesic} if it is an isometry, i.e. \( d(\gamma(t), \gamma(s)) = |t - s| \) for every \( s, t \) in the domain of \( \gamma \), where \( L \) denotes length. We say that \( X \) is a \emph{geodesic metric space} if for every \( x, y \in X \) there exists a geodesic joining \( x \) and \( y \); we denote by \( [x, y] \) any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected. If \( X \) is a graph, we use the notation \([u, v]\) for the edge of a graph joining the vertices \( u \) and \( v \).

In order to consider a graph \( G \) as a geodesic metric space, we must identify any edge \([u, v] \in E(G)\) with the real interval \([0, l]\) (if \( l := L([u, v])\)); hence, if we consider the edge \([u, v]\) as a graph with just one edge, then it is isometric to \([0, l]\). Therefore, any point in the interior of any edge is a point of \( G \).

A connected graph \( G \) is naturally equipped with a distance defined on its points, induced by taking shortest paths in \( G \). Then, we see \( G \) as a metric graph. Throughout this paper we consider graphs which are connected and locally finite (i.e., in each ball there are just a finite number of edges); we allow loops and multiple edges in the graphs; we also allow edges of arbitrary lengths. These conditions guarantee that the graph is a geodesic metric space (since we consider that every point in any edge of a graph \( G \) is a point of \( G \), whether or not it is a vertex of \( G \)).

If \( X \) is a geodesic metric space and \( J = \{J_1, J_2, \ldots, J_n\} \) is a polygon, with sides \( J_i \subseteq X \), we say that \( J \) is \( \delta \)-\emph{thin} if for every \( x \in J_i \) we have that \( d(x, \cup_{j \neq i} J_j) \leq \delta \). We denote by \( \delta(J) \) the sharp thin constant of \( J \), i.e. \( \delta(J) := \inf \{ \delta \geq 0 : J \text{ is } \delta \text{-thin} \} \). If \( x_1, x_2, x_3 \in X \), a \emph{geodesic triangle} \( T = \{x_1, x_2, x_3\} \) is the union of the three geodesics \([x_1 x_2], [x_2 x_3] \) and \([x_3 x_1]\); it is usual to write also \( T = \{[x_1 x_2], [x_2 x_3], [x_3 x_1]\} \). The space \( X \) is \( \delta \)-\emph{hyperbolic} (or satisfies the \emph{Rips condition} with constant \( \delta \)) if every geodesic triangle in \( X \) is \( \delta \)-thin. We denote by \( \delta(X) \) the sharp hyperbolicity constant of \( X \), i.e. \( \delta(X) := \sup \{ \delta(T) : T \text{ is a geodesic triangle in } X \} \). We say that \( X \) is \emph{hyperbolic} if \( X \) is \( \delta \)-hyperbolic for some \( \delta \geq 0 \). If we have a triangle with two identical vertices, we call it a “bigon”. Obviously, every bigon in a \( \delta \)-hyperbolic space is \( \delta \)-thin. It is also clear that every geodesic polygon with \( n \) sides \((n \geq 3)\) in a \( \delta \)-hyperbolic space is \((n - 2)\)-\( \delta \)-thin.

The main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant \( \delta(X) \) of a geodesic metric space can be viewed as a measure of how “tree-like” the space is, since those spaces with \( \delta(X) = 0 \) are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see e.g. [14]).

We would like to point out that deciding whether or not a space is hyperbolic is usually extraordinarily difficult. Note that, first of all, we have to consider an arbitrary geodesic triangle \( T \), and calculate the minimum distance from an arbitrary point \( P \) of \( T \) to the union of the other two sides of the triangle to which \( P \) does not belong to. And then we have to take supremum over all the possible choices for \( P \) and then over all the possible choices for \( T \). Without disregarding the difficulty of solving this minimax problem, note that in general the main obstacle is that we do not know the location of geodesics in the space. Therefore, it is interesting to obtain inequalities involving the hyperbolicity constant of graphs. Since to obtain a characterization of hyperbolic graphs is a very ambitious goal, it is very useful to know hyperbolicity criteria for graphs.

Therefore, one of the main problems on the theory of hyperbolic graphs is to relate the hyperbolicity with other properties on graph theory. In this paper we extend in two ways (edge-chordality and path-chordality) the classical definition of chordal graphs in order to relate this property with Gromov hyperbolicity. In fact, we prove in Section 3 that every edge-chordal graph is hyperbolic (see Theorem 3.4) and that every hyperbolic graph is path-chordal (see Theorem 3.7). Although the converse of these
two Theorems do not hold (see Examples 3.5 and 3.8), the path-chordality is a very close condition to hyperbolicity, in the following sense: in Section 4 we prove that every path-chordal cubic graph (with small path-chordality constant) is hyperbolic (recall that, in order to study Gromov hyperbolicity, general graphs are equivalent to cubic graphs, see Section 2).

2. Background on hyperbolic and cubic graphs

Let \((X, d_X)\) and \((Y, d_Y)\) be two metric spaces. A map \(f : X \to Y\) is said to be an \((\alpha, \beta)\)-quasi-isometric embedding, with constants \(\alpha \geq 1, \beta \geq 0\) if for every \(x, y \in X\):

\[
\alpha^{-1}d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.
\]

The function \(f\) is \(\varepsilon\)-full if for each \(y \in Y\) there exists \(x \in X\) with \(d_Y(f(x), y) \leq \varepsilon\).

A map \(f : X \to Y\) is said to be a quasi-isometry, if there exist constants \(\alpha \geq 1, \beta, \varepsilon \geq 0\) such that \(f\) is an \(\varepsilon\)-full \((\alpha, \beta)\)-quasi-isometric embedding.

Note that a quasi-isometric embedding, in general, is not continuous.

A fundamental property of hyperbolic spaces is the following (see, e.g., [21]).

**Theorem 2.1** (Invariance of hyperbolicity). Let \(f : X \to Y\) be an \((\alpha, \beta)\)-quasi-isometric embedding between the geodesic metric spaces \(X\) and \(Y\). If \(Y\) is hyperbolic, then \(X\) is hyperbolic.

Besides, if \(f\) is \(\varepsilon\)-full for some \(\varepsilon \geq 0\) (a quasi-isometry), then \(X\) is hyperbolic if and only if \(Y\) is hyperbolic.

Furthermore, if \(X\) (respectively, \(Y\)) is \(\delta\)-hyperbolic, then \(Y\) (respectively, \(X\)) is \(\delta'\)-hyperbolic, where \(\delta'\) is a constant which just depends on \(\delta, \alpha, \beta\) and \(\varepsilon\) (respectively, \(\delta, \alpha, \beta\)).

Cubic graphs (graphs with all of their vertices of degree 3) are very interesting in many situations (see, e.g., [7, 9, 17]). Theorem 2.1 and the following results show that they are also very important in the study of Gromov hyperbolicity (see [4, Theorem 21] and [37, Theorem 3.4]).

**Theorem 2.2.** Given any graph \(G\) and any \(\varepsilon > 0\) there exist a cubic graph \(G'\) and an \(\varepsilon\)-full \((1 + \varepsilon, \varepsilon)\) quasi-isometry \(f : G \to G'\).

**Theorem 2.3.** For any graph \(G\) with every edge of length 1 and maximum degree \(\Delta\), there exist a constant \(b\) depending just on \(\Delta\), a cubic graph \(G''\) with every edge of length 1 and a \(b\)-full \((2b, b)\)-quasi-isometry \(f : G \to G''\).

As usual, by cycle in a graph we mean a simple closed curve, i.e., a path with different vertices, except for the last one, which is equal to the first vertex.

In [45, Lemma 2.1] or [4, Corollary 4] we found the following result.

**Lemma 2.4.** In any graph \(G\) we have \(\delta(G) = \sup \{\delta(T) : T \text{ is a geodesic triangle that is a cycle}\}\).

The following result (see [3, Theorem 2.7]) will be useful. Let us denote by \(J(G)\) the set of vertices and midpoints of the edges of \(G\), and by \(T(G)\) the set of geodesic triangles \(T = \{x, y, z\}\) in \(G\) that are cycles with \(x, y, z \in J(G)\).

**Theorem 2.5.** For any graph \(G\) with edges of length 1 we have \(\delta(G) = \sup \{\delta(T) : T \in T(G)\}\). Furthermore, if \(G\) is hyperbolic, there exists a geodesic triangle \(T \in T(G)\) with \(\delta(T) = \delta(G)\).

Let us define the circumference \(c(G)\) of a graph \(G\) as the supremum of the lengths of its cycles if \(G\) is not a tree; we define \(c(G) = 0\) for every tree \(G\). The following results (see [12, Proposition 3.9 and Lemma 3.3]) will be useful.

**Proposition 2.6.** For any graph \(G\)

\[
\delta(G) \leq \frac{1}{4} c(G).
\]
A subgraph $\Gamma$ of $G$ is said isometric if $d_{\Gamma}(x, y) = d_{G}(x, y)$ for every $x, y \in \Gamma$.

**Lemma 2.7.** If $G$ is any graph, then
$$\delta(G) \geq \frac{1}{4} \sup \{L(g) : g \text{ is an isometric cycle in } G\}.$$  

We will need the following result (see [11, Theorem 3.5]).

**Theorem 2.8.** Suppose that a graph $G$ is the 1-skeleton of a tesselation of $\mathbb{R}^2$ with convex tiles $\{F_n\}$. If $\inf_n A(F_n) > 0$, then $G$ is not hyperbolic.

### 3. Edge-chordal and path-chordal graphs

**Definition 3.1.** A shortcut in a cycle $C$ of a graph $G$ is a path $\sigma$ joining two vertices $p, q \in C$ such that $L(\sigma) < d_G(p, q)$.

An edge-shortcut in a cycle $C$ is an edge of $G$ which is a shortcut of $C$.

Given two constants $k, m \geq 0$, we say that a graph $G$ is $(k, m)$-edge-chordal if for any cycle $C$ in $G$ with length $L(C) \geq k$ there exists an edge-shortcut $e$ with length $L(e) \leq m$. The graph $G$ is edge-chordal if there exist constants $k, m \geq 0$ such that $G$ is $(k, m)$-edge-chordal.

We say that a graph $G$ is $r$-path-chordal if in every cycle $C$ in $G$ with $L(C) \geq r$ there exists at least a shortcut $\sigma$ with $L(\sigma) \leq r/2$.

Note that every $(k, m)$-edge-chordal graph is max$\{k, 2m\}$-path-chordal.

Usually a graph (with edges of length 1) is said chordal if it is $(4, 1)$-edge-chordal according to Definition 3.1. In [8] the authors prove that chordal graphs are hyperbolic. In [49] the authors introduce $k$-chordal graphs generalizing the chordality (a graph with edges of length 1 is $k$-chordal if it does not contain any induced $n$-cycle for $n > k$; then chordal graphs are 3-chordal) and they prove that $k$-chordal graphs are hyperbolic. Our concept of edge-chordality generalizes the $k$-chordality; in fact, $k$-chordal graphs are $(k + 1, 1)$-edge-chordal. We prove in this Section that every edge-chordal graph is hyperbolic (see Theorem 3.4) and that every hyperbolic graph is path-chordal (see Theorem 3.7).

We need some previous lemmas.

**Lemma 3.2.** Given a $(k, m)$-edge-chordal graph $G$, a cycle $C$ in $G$ with length $L(C) \geq k$ and a geodesic $[ab] \subset C$ with $L([ab]) \geq k/2$, there exist two vertices $v \in V(G) \cap ([ab] \setminus \{a, b\})$ and $w \in V(G) \cap (C \setminus [ab])$ with $e = [v, w] \in E(G)$, $L(e) < d_G(v, w)$ and $L(e) \leq m$.

**Proof.** Since $[ab]$ is a geodesic contained in $C$, we have $L(C \setminus [ab]) \geq L([ab])$ and $L(C) \geq 2L([ab])$.

Assume first that $L(C) = 2L([ab])$. In this case $L(C \setminus [ab]) = L([ab])$ and then $(C \setminus [ab]) \cup \{a, b\}$ is also a geodesic joining $a$ and $b$. Since $L(C) \geq k$ and $G$ is a $(k, m)$-edge-chordal graph, there exists an edge $e = [x, y]$ with $x, y \in V(G) \cap C$ such that $L(e) < d_G(x, y)$ and $L(e) \leq m$. It is not possible for $e$ to join two vertices of $[ab]$, since $[ab]$ is a geodesic. Similarly, it is not possible for $e$ to join two vertices of $(C \setminus [ab]) \cup \{a, b\}$, since $(C \setminus [ab]) \cup \{a, b\}$ is also a geodesic. Therefore, the conclusion of the lemma holds in this case.

Assume now that $L(C) > 2L([ab])$. Since $L(C) \geq k$ and $G$ is a $(k, m)$-edge-chordal graph, there exists an edge $e = [x, y]$ with $x, y \in V(G) \cap C$ such that $L(e) < d_G(x, y)$ and $L(e) \leq m$. It is not possible for $e$ to join two vertices of $[ab]$, since $[ab]$ is a geodesic. If either $x$ or $y$ belongs to $[ab] \setminus \{a, b\}$, then the conclusion of the lemma also holds in this case. If $x, y \notin [ab] \setminus \{a, b\}$, then we consider the cycle $C_1$ obtained by pasting $e$ with the connected component of $C \setminus \{x, y\}$ which contains $[ab] \setminus \{a, b\}$. It is clear that $L(C_1) < L(C)$, $[ab] \subset C_1$ and $V(G) \cap C_1 \subset V(G) \cap C$.

Now we can apply the previous argument to $C_1$. If we do not obtain the conclusion of the Lemma, then we obtain a new cycle $C_2$ with $L(C_2) < L(C_1) < L(C)$, $[ab] \subset C_2$ and $V(G) \cap C_2 \subset$
V(G) ∩ C. Iterating this process we obtain either the conclusion of the Lemma or a sequence of cycles C_1, C_2, . . . , C_j, . . . with [ab] ⊂ C_j, V(G) ∩ C_j ⊆ V(G) ∩ C for every j ≥ 1, and

\[ L(C_j) < \cdots < L(C_2) < L(C_1) < L(C). \]

Since G is locally finite and we have (3.3), this process must stop in some finite step by compactness; therefore, the conclusion of the Lemma holds. □

The following results give a necessary and a sufficient condition (which are close) for the hyperbolicity of graphs. In fact, we prove that edge-chordality implies hyperbolicity and that hyperbolicity implies path-chordality.

**Theorem 3.4.** If G is a (k, m)-edge-chordal graph, then it is (m + k/4)-hyperbolic.

**Proof.** Let us consider any fixed geodesic triangle T = \{x, y, z\} in G. By Lemma 2.4, in order to compute δ(T) we can assume that T is a cycle. Without loss of generality we can assume also that there exists p ∈ [xy] with δ(T) = d(p, [xz] ∪ [yz]). If δ(T) ≤ k/4, then there is nothing to prove. If δ(T) > k/4, then d(p, \{x, y\}) > k/4, L([xy]) > k/2 and L(T) > k. Let us consider a, b ∈ [xy], with a ≠ b and d(a, p) = d(b, p) = k/4; then p ∈ [ab] ⊂ [xy] and L([ab]) = k/2. Lemma 3.2 gives that there exist two vertices v ∈ V(G) ∩ (\{ab\} \ {a, b}) and w ∈ V(G) ∩ (T \ \{ab\}) with v = [v, w] ∈ E(G), L(e) < d_T(v, w) and L(e) ≤ m. Note that w ∉ [xy] since L([v, w]) < d_T(v, w). Therefore,

\[ d(p, [xz] \cup [yz]) ≤ d(p, w) ≤ d(p, v) + d(v, w) ≤ \frac{k}{4} + m. \]

Then G is (m + k/4)-hyperbolic. □

The following example shows that the converse of Theorem 3.4 does not hold, i.e., hyperbolicity does not imply edge-chordality.

**Example 3.5.** Let P_3 be the path graph with (adjacent) vertices v_1, v_2, v_3, and G the Cartesian product graph G = Z × P_3 with L(e) = 1 for every e ∈ E(G). Since G and Z are quasi-isometric, G is hyperbolic. One can check that G is 5-path-chordal, but it is not edge-chordal, since for every natural number r ≥ 2 the geodesic squares with vertices (0, v_1), (r, v_1), (r, v_3), (0, v_3) do not have edge-shortcuts.

Theorem 3.7 below is a kind of converse of Theorem 3.4. In order to prove it, we need the following technical result.

**Theorem 3.6.** ([21, p.92]) Let us consider constants δ ≥ 0, r > 0, a δ-hyperbolic geodesic metric space X and a finite sequence \{x_j\}_{0 ≤ j ≤ n} in X with

\[ d_X(x_{j-1}, x_{j+1}) ≥ \max\{d_X(x_{j-1}, x_j), d_X(x_j, x_{j+1})\} + 18δ + r \]

for every 0 < j < n. Then d_X(x_0, x_n) ≥ rn.

**Theorem 3.7.** Every δ-hyperbolic graph is 90δ-path-chordal.

**Proof.** Seeking for a contradiction, assume that G is a δ-hyperbolic graph which is not 90δ-path-chordal. Then there exists a cycle C in G with L(C) ≥ 90δ without shortcuts σ with L(σ) ≤ 45δ. Consequently, any subcurve g of C with L(g) ≤ 45δ is a geodesic in G. Let us define an integer n and a positive number ℓ by

\[ n := \left\lceil \frac{2L(C)}{45δ} \right\rceil, \quad ℓ := \frac{L(C)}{n}. \]

Since

\[ \frac{2L(C)}{45δ} ≤ \left\lceil \frac{2L(C)}{45δ} \right\rceil < \frac{2L(C)}{45δ} + 1, \]

we deduce that

\[ 18δ < ℓ ≤ \frac{45δ}{2}. \]
Choose a finite sequence \( \{x_j\}_{0 \leq j \leq n} \) in \( C \) with \( d_X(x_j, x_{j+1}) = d_C(x_j, x_{j+1}) = \ell \) for every \( 0 \leq j < n \), and \( d_X(x_j, x_{j+1}) = 2 d_C(x_j, x_{j+1}) = 2\ell \) for every \( 0 < j < n \); then \( x_0 = x_n \).

If we define \( r := \ell - 18\delta \), then
\[
2\ell = \ell + 18\delta + r,
\]
\[
d_X(x_{j-1}, x_{j+1}) = \max\{d_X(x_{j-1}, x_j), d_X(x_j, x_{j+1})\} + 18\delta + r,
\]
for every \( 0 < j < n \). Then Theorem 3.6 gives \( 0 = d_X(x_0, x_n) \geq rn > 0 \), which is the contradiction we were looking for; hence, we conclude that \( G \) is \( 90\delta \)-path-chordal.

The following example shows that the converse of Theorem 3.7 does not hold, i.e., path-chordality does not imply hyperbolicity.

**Example 3.8.** First of all, we assume that \( 0 \in \mathbb{N} \). Let \( \sum_{n=0}^{\infty} a_n \) be a fixed convergent series of positive real numbers such that \( a_0 = 1 \) and \( \sum_{n=0}^{\infty} a_n = S < \infty \). Now, Let us consider \( \{S_n\}_{n=0}^{\infty} \) the sequence of partial sums. Let \( G \) be the Cartesian product graph \( G = \mathbb{N} \times \mathbb{N} \) with \( L((p, q), (p+1, q)) = S_{p+q} = L((p, q), (p, q+1)) \).

Note that \( G \) is a path chordal graph, since each cycle \( C \) of \( G \) with \( L(C) > 4S \) has a vertex \( v = (p+1, q+1) \) in \( C \) such that \( [(p+1, q), v], [(p, q+1), v] \in E(G) \) are contained in \( C \) (i.e., \( v \) is an upper-right vertex of \( C \)); then \( C \) has a shortcut \( \sigma \subset [(p+1, q), (p, q)] \cup [(p, q), (p, q+1)] \), thus \( S_{p+q} < S_{p+q+1} \).

Let \( G_0 \) be the Cartesian product graph \( G = \mathbb{N} \times \mathbb{N} \) with \( L(e) = 1 \) for every \( e \in E(G_0) \). Since \( G_0 \) and \( G \) are quasi-isometric, \( G \) is not hyperbolic.

### 4. Chordality in Cubic Graphs

We want to remark that by Theorems 2.1, 2.2 and 2.3, the study of the hyperbolicity of graphs can be reduced to the study of the hyperbolicity of cubic graphs. Along this Section we just consider (finite or infinite) graphs with edges of length 1.

In this section we obtain several results which guarantee the hyperbolicity of many path-chordal cubic graphs (see Theorems 4.4 and 4.9).

A proper shortcut in \( C \) is a shortcut \( \sigma \) joining two vertices \( p, q \in C \cap V(G) \) such that \( \sigma \cap C = \{p, q\} \) and \( \sigma \) is a geodesic. Note that in any cycle \( C \) of a \( r \)-path-chordal graph \( G \) such that \( L(C) \geq r \) there is a proper shortcut with length at least \( r/2 \). Therefore, we may replace proper shortcut by shortcut in the definition of chordal graph.

Note that, since we just consider graphs with edges of length 1, every edge-shortcut is a proper shortcut.

**Theorem 4.1.** Let \( G \) be any cubic graph. Then \( G \) is 4-path-chordal if and only if it is a chordal.

**Proof.** If \( G \) is a chordal graph, then it is 4-path-chordal.

Assume now that \( G \) is a 4-path-chordal graph. Seeking for a contradiction, assume that there exists a cycle \( C \) in \( G \) with \( L(C) \geq 4 \) and such that \( C \) has no shortcut with length 1. Since \( L(C) \geq 4 \) and \( G \) is 4-path-chordal, the set \( V_C := \{u, v \mid u, v \in V(G) \cap C \text{ and } [u, v] \text{ is a shortcut in } C \text{ with length 2} \} \) is non-empty. Let \( (x, y) \in V_C \) with \( d_C(x, y) = \min\{d_C(u, v) \mid (u, v) \in V_C\} \). Let \( g_1 \) be a path joining \( x \) and \( y \) contained in \( C \) such that \( L(g_1) = d_C(x, y) \). Define \( C_1 := g_1 \cup [xy] \); then \( L(C_1) \geq 2L([xy]) \geq 4 \) and there exists a proper shortcut \( \rho = [zw] \) in \( C_1 \). Since it is not possible to have \( \{z, w\} \subset [xy] \) or \( \{z, w\} \subset g_1 \), without loss of generality we can assume that \( z \in g_1 \setminus \{x, y\} \) and \( w \in [xy] \setminus \{x, y\} \); since \( L([xy]) = 2 \), \( w \) is the midpoint of \([xy]\).

Note that we have either \( L(\rho) = 1 \) or \( L(\rho) = 2 \).

If \( L(\rho) = 1 \), then \( d_C(z, x) \leq 2 \) and \( d_C(z, y) \leq 2 \), since \( [x, w] \cup [w, z] \) and \( [y, w] \cup [w, z] \) are not shortcuts in \( C \). We prove now that \( d_C(z, x) = d_C(z, y) = 1 \). Otherwise, by symmetry, we can assume that \( d_C(z, x) = 2 \); then the cycle \( C_2 = [x, w] \cup [w, z] \) has length 4 and there exists a shortcut in \( C_2 \); but since \( x, z, w \) have “full degree”, there is just one vertex in \( C_2 \) that can be an endpoint of the shortcut.
This is a contradiction and we conclude that $dc(z, x) = dc(z, y) = 1$. Then $dc(x, y) = 2 = L([xy])$ and $[xy]$ is not a shortcut in $C$, which is a contradiction.

If $L(p) = 2$, then we have a shortcut in each of the two induced cycles on $C_1$ by $p$; if $v$ is the midpoint of $p$, then $v$ is an endpoint of the two shortcuts. Since $G$ is a cubic graph, the two shortcuts are $[v, v_0] \cup [v_0, v_1]$ and $[v, v_0] \cup [v_0, v_2]$ for some vertices $v_0 \in V(G)$ and $v_1, v_2 \in V(G) \cap g_1$. If $g_2$ is the path contained in $g_1$, joining $v_1$ and $v_2$, then $\gamma = [v_1, v_0] \cup [v_0, v_2] \cup g_2$ is a cycle with $L(\gamma) \geq 4$. Since $\gamma$ does not have a shortcut, we obtain a contradiction.

Hence, we conclude that $G$ is a chordal graph.

\[ \square \]

**Lemma 4.2.** Let $G$ be a 4-path-chordal cubic graph and let $C$ be any cycle in $G$ with two different shortcuts with length 1. Then, $G$ is isomorphic to the complete graph with 4 vertices $K_4$.

**Proof.** By Theorem 4.1 any cycle of $G$ with length greater than 3 has an edge-shortcut. Let $\sigma_1 := [x, x']$ and $\sigma_2 := [y, y']$ be two different edge-shortcuts in $C$. Let $g$ (respectively, $g'$) be a subcurve of $C$ joining $x$ and $y$ (respectively, $x'$ and $y'$) such that $g \cap g' = \emptyset$; then $C_1 := \sigma_1 \cup g' \cup \sigma_2 \cup g$ is a cycle with $L(C_1) \geq 4$. The cycle $C$ can be oriented either by: (1) $x \rightarrow y \rightarrow y' \rightarrow x'$, or (2) $x \rightarrow y \rightarrow x' \rightarrow y'$.

Assume that $C$ is oriented by (1). Then $C_1$ has an edge-shortcut $e_1$ joining $y \setminus \{x, y\}$ and $g' \setminus \{x', y'\}$. Let $C_2$ be a cycle obtained by joining $e_1$ with a path contained in $C_1$. Proceeding this way, we obtain a finite sequence of cycles $C_1, C_2, \ldots, C_k$ such that $L(C) > L(C_1) > L(C_2) > \cdots > L(C_k) = 4$ and the four vertices of $C_k$ have full degree; then there is no shortcut in $C_k$, which is a contradiction.

Assume now that $C$ is oriented by (2). Let $\gamma_1, \gamma_2$ be two curves with $\gamma_1 \cup \gamma_2 = C$ and $\gamma_1 \cap \gamma_2 = \{x, x'\}$. If $\max\{L(\gamma_1), L(\gamma_2)\} > 2$, then without loss of generality we can assume that $L(\gamma_1) > 2$; hence, $\gamma_1 \cup \{x, x'\}$ is a cycle with $L(\gamma_1 \cup \{x, x'\}) \geq 4$ and there is an edge-shortcut $e_1$ in $\gamma_1 \cup \{x, x'\}$; since $x$ and $x'$ have full degree, $\{x, x'\} \cap e_1 = \emptyset$; consequently, $\{x, x'\}$ and $e_1$ are two edge-shortcuts in $C$ in the case (1), and we have proved that this is a contradiction. Therefore, $\max\{L(\gamma_1), L(\gamma_2)\} \leq 2$; we conclude that $L(\gamma_1) = L(\gamma_2) = 2$, and then $G$ is isomorphic to $K_4$.

\[ \square \]

**Corollary 4.3.** If $G$ is a 4-path-chordal cubic graph, then $G$ does not have cycles with length greater than 4.

**Proof.** Seeking for a contradiction, assume that there exists a cycle $C$ with length $r > 4$.

If $r = 5$, then there is an edge-shortcut $[x, y]$ with $dc(x, y) = 2$. If $g$ is the path in $C$ joining $x$ and $y$ with length 3, then $[x, y] \cup g$ is a cycle with length 4 and there is no shortcut in it since $x$ and $y$ have full degree. This is the contradiction we were looking for.

If $r > 5$, then there is an edge-shortcut $[x, y]$. Let $g_1, g_2$ be two paths with $g_1 \cup g_2 = C$ and $g_1 \cap g_2 = \{x, y\}$. Without loss of generality we can assume that $L(g_1) \geq L(g_2)$; then $[x, y] \cup g_1$ is a cycle with $L([x, y] \cup g_1) \geq 4$ and there exists an edge-shortcut $e$ in $[x, y] \cup g_1$. Hence, $[x, y]$ and $e$ are two edge-shortcuts in $C$, and $G$ is isometric to $K_4$ by Lemma 4.2. This is a contradiction.

\[ \square \]

Corollary 4.3 and Proposition 2.6 have the following consequence.

**Theorem 4.4.** If $G$ is a 4-path-chordal cubic graph, then $G$ is 1-hyperbolic.

The following result provides a simple and explicit formula for the hyperbolicity constant of the 4-path-chordal cubic graphs.

**Theorem 4.5.** If $G$ is a 4-path-chordal cubic graph, then $\delta(G) = c(G)/4$.

**Proof.** By Proposition 2.6, $\delta(G) \leq c(G)/4$. Let us prove the converse inequality. By Corollary 4.3 we have $c(G) \leq 4$. If $c(G) \leq 3$, then $\delta(G) \geq c(G)/4$ by Lemma 2.7. Assume now that $c(G) = 4$ and consider a cycle $g$ with length 4. Let $x, y$ be midpoints of edges in $g$ with $d(x, y) = 2$ and paths $g_1, g_2$ with $g_1 \cup g_2 = g$ and $g_1 \cap g_2 = \{x, y\}$. Then $\{g_1, g_2\}$ is a geodesic bigon in $G$. If $p$ is the midpoint of $g_1$, then $\delta(G) \geq d(p, g_2) = d(p, \{x, y\}) = 1 = c(G)/4$.

\[ \square \]
Recall that given an edge $e = [u, v]$ in a graph $G$ the edge contraction of $G$ (relative to $e$) is the graph obtained as follows: the edge $e$ is removed and its two incident vertices are merged into a new vertex $w$, where the edges incident to $w$ each correspond to an edge incident to either $u$ or $v$. The following result characterizes in a simple and precise way the 4-path-chordal cubic graphs.

**Theorem 4.6.** $G$ is a 4-path-chordal cubic graph if and only if $G$ is isomorphic to one of the following graphs:

1. a complete graph with 4 vertices $K_4$.
2. a graph with exactly 2 vertices and a 3-multiple edge joining them.
3. a graph obtained from any tree with vertices of degree at most 3 such that we replace
   - each vertex of degree 1 by a loop or a cycle graph with 3 vertices and a double edge,
   - each vertex of degree 2 by a complete graph with 4 vertices without one edge, or a graph
     with two vertices and two multiple edges,
   - each vertex in an arbitrary subset of vertices with degree 3 by a cycle graph $C_3$.

**Proof.** Assume that $G$ is a 4-path-chordal cubic graph. If $G$ is isomorphic to the graphs in (1) or (2), then we have finished. Assume now that $G$ is not isomorphic to the graphs in (1) or (2). By Corollary 4.3 we have $c(G) \leq 4$. Since $G$ is a 4-path-chordal cubic graph, the cycles with length 4 are pairwise disjoint, and the induced graph by the vertices of each cycle with length 4 is isomorphic to a complete graph with 4 vertices without one edge.

Let $G_1$ be the graph obtained by the contraction of every edge in every cycle with length 4; then $G_1$ has a vertex of degree 2 corresponding to each cycle in $G$ with length 4. Since $G_1$ is a graph with vertices of degree at most 3 and $c(G) \leq 3$, its cycles with length 3 are pairwise disjoint (except the cycles with different edges of the same double edge).

Let $G_2$ be the graph obtained by the contraction of every edge in every cycle with length 3; then $G_2$ has a vertex of degree 1 corresponding to each cycle graph with 3 vertices and a double edge, and a vertex of degree 3 corresponding to each cycle graph with 3 vertices and simple edges. Since $G_2$ is a graph with vertices of degree at most 3 and $c(G) \leq 2$, its cycles with length 1 or 2 are pairwise disjoint.

Let $G_3$ be the graph obtained by the contraction of every double edge and every loop; then $G_3$ has a vertex of degree 1 corresponding to each loop, and a vertex of degree 2 corresponding to each double edge. Then $G_3$ is a tree with vertices of degree at most 3.

One can check easily the converse implication. \qed

**Lemma 4.7.** If $G$ is a 5-path-chordal cubic graph, then there are no proper shortcuts with length 2 in any cycle of $G$.

**Proof.** We prove the Lemma by complete induction. It is clear that on every cycle in $G$ with length 5 the proper shortcuts have length 1. Now, we assume that any cycle in $G$ with length at most $k$ does not have proper shortcuts with length 2.

Consider a cycle $C$ in $G$ with $k+1$ vertices. Seeking for a contradiction, assume that $C$ has a proper shortcut $\sigma := [xy]$ with length 2, and let $v$ be the midpoint of $\sigma$. Let $g_1, g_2$ be two paths in $G$ joining $x$ and $y$ such that $C = g_1 \cup g_2$ and $g_1 \cap g_2 = \{x, y\}$. Consider the cycles $C_1 := g_1 \cup \sigma$ and $C_2 := g_2 \cup \sigma$. Let $\rho_1$ be a proper shortcut in $C_1$; by hypothesis, $L(\rho_1) = 1$. If $\rho_1$ joins two vertices $u$ and $v$ in $g_1$, then denote by $g'_1$ the path joining $u, v$ contained in $C$ and which contains $g_2$; the cycle $\rho_1 \cup g'_1$ verifies $L(\rho_1 \cup g'_1) \leq k$ and has the proper shortcut $\sigma$ with length 2, which is a contradiction. Hence, $\rho_1$ does not join two vertices in $g_1$. Since $x$ and $y$ have full degree, $\rho_1 = [v, z]$ with $z \in g_1 \setminus \{x, y\}$. In a similar way, there exists another shortcut $[v, w]$ with $w \in g_2 \setminus \{x, y\}$. Hence, $\deg(v) \geq 4$; this is the contradiction we were looking for and we conclude that $C$ does not have proper shortcuts with length 2. \qed
By Lemma 4.7 any 5-path-chordal cubic graph $G$ is $(5,1)$-edge-chordal, and Theorem 3.4 gives that $\delta(G) \le 9/4$. However, Theorem 4.9 below improves this inequality.

We need the following technical result.

**Lemma 4.8.** Let $C$ be a cycle in a 5-path-chordal cubic graph $G$ and $[xy]$ a geodesic contained in $C$. If there are two edge-shortcuts $\rho_1 := [x,u], \rho_2 := [y,v]$ in $C$ and there is no other edge-shortcut in $C$ starting in $[xy]$, then $[x,y], [u,v] \in E(G)$.

Furthermore, the cycle obtained by joining the shortcuts $\rho_1$ and $\rho_2$ with paths contained in $C$ has length 4.

**Proof.** Denote by $\gamma$ the path contained in $C$ which joins $u$ and $v$ such that $x, y \notin \gamma$. Denote by $C_1$ the cycle $C_1 := \rho_1 \cup [xy] \cup \rho_2 \cup \gamma$. Notice that is suffices to prove that $L(C_1) = 4$. Seeking for a contradiction, assume that $L(C_1) > 4$. Then, there is an edge-shortcut $\sigma := [u_1,v_1]$ in $C_1$ joining two points of $\gamma$ such that $d_\gamma(u_1,v_1)$ is maximum. Without loss of generality we can suppose that $\gamma$ can be oriented by $u \to u_1 \to v_1 \to v$. Since $G$ is a cubic graph, we have $\rho_1 \cap \sigma = \emptyset$ for $i \in \{1,2\}$; then we have that the cycle $C_2 := \rho_1 \cup [u_1,v_1] \cup \sigma \cup [v_1,v] \cup \rho_2 \cup [xy]$ has length greater than 5; since $C_2$ does not have edge-shortcuts, we obtain the contradiction we were looking for. Therefore, we conclude that $L(C_1) = 4$ and $[x,y], [u,v] \in E(G)$.

**Theorem 4.9.** If $G$ is a 5-path-chordal cubic graph, then $G$ is $(3/2)$-hyperbolic.

**Proof.** Fix a geodesic triangle $T = \{x,y,z\}$ in $G$. By Theorem 2.5, in order to study $\delta(G)$ we can assume that $T$ is a cycle with $x, y, z \in J(G)$. If $L(T) \le 6$, then the three geodesic sides of $T$ have length at most 3 and, consequently, $\delta(T) \le 3/2$. Assume now that $L(T) \ge 7$. By Lemma 4.7 there exists an edge-shortcut in $T$. By symmetry, it suffices to prove that for every $p \in [xy]$ we have $d_G(p,[yz] \cup [zx]) \le 3/2$.

Assume first that there is no edge-shortcut in $T$ starting in $[xy]$. Since $G$ is $(5,1)$-edge-chordal, by Lemma 3.2 we have that $L([xy]) \le 2$; therefore, we have for every $p \in [xy]$,

$$d_G(p,[yz] \cup [zx]) \le \frac{1}{2}.$$ 

Assume now that there is an edge-shortcut in $T$ joining $[xy]$ and $[zx]$, but there is no edge-shortcut joining $[xy]$ and $[yz]$. Let $\sigma_1$ be an edge-shortcut in $T$ joining $P_1$ and $Q_1$, where $P_1 \in [xy], Q_1 \in [xz]$ and $P_1$ is the closest vertex to $x$ with an edge-shortcut. Consider the cycle $C := [xP_1] \cup \sigma_1 \cup [Q_1x]$. Then, $C$ does not have edge-shortcuts; therefore, $L(C) \le 4$ and $L([xP_1]) + L([Q_1x]) \le 3$. Hence, since $L([xP_1]) \le 1 + L([xQ_1]),$ we have $L([xP_1]) \le 2, L([xP_1]) + L([Q_1x]) \le 3$ and we obtain $d_G(p,[yz] \cup [zx]) \le \frac{1}{2} = d_G(p,[xz]) \le \frac{1}{2} = d_G(p,[xQ_1]) \le \frac{1}{2}$ for every $p \in [xP_1]$. Let $n$ be the exact number of edge-shortcuts in $T$ joining $[xy]$ and $[zx]$. Let $P_1, \ldots, P_n \in [xy], Q_1, \ldots, Q_n \in [xz]$ with $[P_i, Q_i] \in E(G)$ for $1 \le i \le n$ and $L([xP_1]) < L([xP_{i+1}])$ for $1 \le i < n$. Hence, by Lemma 4.8 we have that $[P_i, P_{i+1}], [Q_i, Q_{i+1}] \in E(G)$ for every $1 \le i < n$; thus, for every $p \in [P_nP_{i+1}],$ we obtain $d_G(p,[yz] \cup [zx]) \le d_G(p,[P_nQ_{i+1}]) \le \frac{1}{2}$.

Furthermore, since there is no edge-shortcut in $T$ from $[P_ny],$ by Lemma 3.2 we have that $L([P_ny]) \le 2$; therefore, for every $p \in [P_ny]$ we have $d_G(p,[yz] \cup [zx]) \le \frac{1}{2} = d_G(p,[Q_ny]) \le \frac{1}{2}$. Hence, we obtain

$$d_G(p,[yz] \cup [zx]) \le \frac{1}{2}, \quad \text{for every } p \in [xy].$$

Finally, assume that there are shortcuts in $T$ joining $[xy]$ with $[xz]$, and $[xy]$ with $[yz]$. Let $m$ be the exact number of edge-shortcuts in $T$ joining $[xy]$ and $[yz]$. Let $R_1, \ldots, R_m \in [xy], S_1, \ldots, S_m \in [yz]$ with $[R_i, S_i] \in E(G)$ for $1 \le i \le m$ and $L(yR_1) < L(yR_{i+1})$ for $1 \le i < m$. Let $1 \le k \le m$ with $[P_nR_k] \cap [R_1, \ldots, R_m] = R_k$; by Lemma 4.8 we have that $[P_nR_k]$ is an edge. So, a similar argument to the one in the previous case gives

$$d_G(p,[yz] \cup [zx]) \le \frac{1}{2}, \quad \text{for every } p \in [xy].$$

□
The equality in Theorem 4.9 is attained by the Cartesian product graphs $P_2 \square P_n$ for $n \geq 4$ with the appropriated addition of two multiple edges.

The following example shows that the converse of Theorem 3.7 does not hold even for cubic graphs, i.e., path-chordality does not imply hyperbolicity in cubic graphs.

**Example 4.10.** Consider a graph $G$ which is the 1-skeleton of the semiregular tessellation of the plane obtained by octagons and squares, see Figure 1.

![Figure 1. Semiregular tessellation of $\mathbb{R}^2$ whose it 1-skeleton is a cubic 18-path-chordal graph.](image)

Clearly, $G$ is a cubic graph; we show now that $G$ is a 18-path-chordal graph. Let us consider a cycle $C$ in $G$ with length greater than 17. Let $R_C$ be the compact region in $\mathbb{R}^2$ whose boundary is $C$. We pay attention to the relative position of the octagons contained in $R_C$. Notice that we have either:

1. there is an octagon $E$ in $R_C$ intersecting $C$ such that either (a) neither of the two octagons which are horizontal neighbors of $E$ are contained in $R_C$ or (b) neither of the two octagons which are vertical neighbors of $E$ are contained in $R_C$ or (c) both of the above, simultaneously,
2. there are three octagons in $R_C$ intersecting $C$ to form a “right angle” (i.e., there is a octagon $E$ in $R_C$ intersecting $C$, the “corner”, such that one of the octagons which are horizontal neighbors of $E$ is contained in $R_C$ and the other one is not contained in $R_C$, and one of the octagons which are vertical neighbors of $E$ is contained in $R_C$ and the other one is not contained in $R_C$) and (1) does not hold,
3. there are four octagons in $R_C$ intersecting $C$ to form a “right angle without the corner”, and (1) and (2) do not hold.

If (1) holds, then $E \setminus C$ is a shortcut in $C$ with length at most 3. If (2) holds, then $C$ has a shortcut of length at most 5 (delimiting the octagon at the corner). If (3) holds, then $C$ has a shortcut of length at most 9 (delimiting the two octagons closest to the corner). This prove that $G$ is 18-path-chordal. Finally, by Theorem 2.8 we have that $G$ is not hyperbolic.
References


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