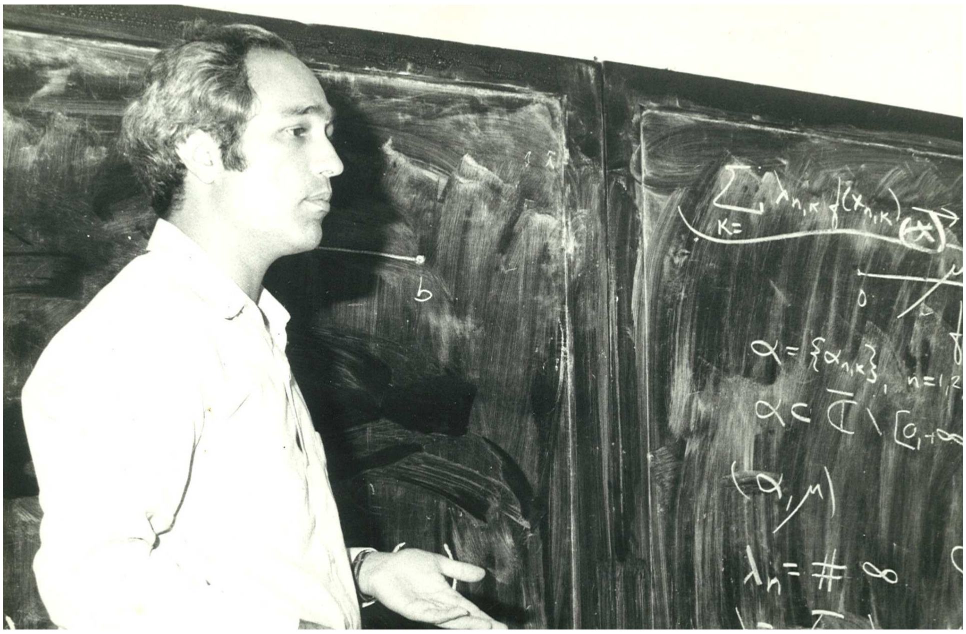
# Guillermo López Lagomasino

# Mathematical Life and Works



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AND APPROXIMATION THEORY

Conference in honor of Guillermo López Lagomasino's 60th birthday

# Guillermo López Lagomasino

# Mathematical Life and Works


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# Contents

# 1 Biographical notes.

Guillermo López Lagomasino was born in Havana, Cuba, on December 21, 1948. His father was an electrician and his mother a schoolteacher. He was a single child. At the age of eight, his family emigrated to United States.
He attended St. Stephen's Primary School in Cleveland, Ohio. There, he also learned to play the violin. During 7th and 8th grade he was concert master of school's orchestra and fourth in the line-up of its baseball team.

At the end of the academic year of 1962 Guillermo's family planned to return to Cuba. As a result of the Cuban Missile Crisis his stay in the USA was extended for another six months. So, he began Central High School in Paterson, New Jersey.

In March, 1963, Guillermo returned to Cuba. As a curious anecdote, he flew back on a direct flight from Miami to La Habana on a Pan American carrier with only four passengers (Guillermo's parents, himself, and the son of a former army coronel of the deposed regime). The flight was carrying food and medicine as part of the indemnity the US agreed to pay the Cuban government for the release of prisoners seized during the attack of Playa Girón. The trip was arranged by the International Red Cross for humanitarian reasons.

Back in Cuba, he continued his studies at Secundaria Básica Carlos de la Torre winning a scholarship for talented students to study at Preuniversitario Cepero Bonilla where he obtained top marks. As a result, he received several awards and distinctions.

Guillermo obtained a degree in Mathematics at Universidad de La Habana where he studied from 1967 to 1971. Along with his studies, he was very active in the University Student's Federation (FEU) where he was elected member of the Universities' Bureau and President of the FEU of the Facultad de Ciencias during his last two academic years. He achieved excellent grades which allowed him, after graduation, to obtain a position as professor in the Departamento de Matemáticas at Facultad de Ciencias of Universidad de La Habana.

After two years working in that faculty, he embarked for the Soviet Union to carry out postgraduate studies. He defended his Ph. D. thesis under the supervision of Academician Andrei A. Gonchar, at Moscow State University in March, 1978. The title of his Memoir was "On the Convergence of Multipoint Padé Approximants." All his research thereafter has been strongly influenced by the Russian School on rational approximation and orthogonal polynomials.

On his return to Havana, he was appointed, first, Assistant Professor (1978–1983), and later, Full Professor (1983–1995). For several years, he was head of the Departamento de Teoría de Funciones and vice-Dean for research of the Faculty of Mathematics. During these years Guillermo develops an intensive work leading the mathematical research in Cuba. He was Editor in Chief and Executive Secretary of the journal "Revista Ciencias Matemáticas" since its foundation in 1980 until 1995. Also Guillermo was chosen for the Executive Council of the Cuban Mathematical Society (La Sociedad Cubana de Matemática y Computación.) Together with Professor Miguel A. Jiménez led an active weekly seminar where the discussions and talks were very instructive for their research students.

He took leave from his responsibilities to visit during 1987–1989 the Steklov Mathematical Institute, USSR, as a Visiting Research Fellow. During his visit he defended a second doctoral thesis entitled "Rational Approximation of Meromorphic Functions of Stieltjes Type". This work deals with multipoint Padé approximants for perturbed Stieltjes functions, orthogonal polynomials with respect to varying measure and its applications. Among other things, he solved a conjecture posed by A.A. Gonchar 10 years earlier. Guillermo showed that orthogonal polynomials with respect to varying measures are an effective tool not only for solving problems on multipoint rational approximation but also for studying questions on orthogonal polynomials involving fixed measures.

In 1991–1992 he spent a sabbatical year at Escuela Técnica Superior de Ingenieros Industriales, Universidad Politécnica de Madrid, Spain, sponsored by the Spanish Ministry of Education and one semester at Universidad Carlos III de Madrid, Spain, 1992–1993. His first long visit to Spain was very productive beginning a fruitful scientific collaboration with Professor Francisco Marcellán, who was his counterpart in the grant, and the Spanish school of orthogonal polynomials.

Guillermo was honored with several distinctions such as: "Forjadores del Futuro, 1981," awarded to young researchers that stand out at a national level in Cuba; the "Award Pablo Miquel, 1984," given by the Cuban Mathematical Society for his overall work; and a "Special Distinction, 1986" given by the Minister of Education to the best researchers among university professors. In 1988, his results were distinguished between the three most valuable of the Steklov Institute that year. Moreover, in 1994 he obtained the "National Order Carlos J. Finlay" (this medal, given by the State Council of Cuba to the best scientists of the country, never reached his hands since he was unable to attend the ceremony because he was already living abroad).

He left Cuba to work at Universidad Carlos III de Madrid in 1995. During 6 years Guillermo was a "permanent" Visiting Scholar. Having obtained the Spanish citizenship in 2000 he applied and obtained, first, a permanent position as Associate Professor in 2001 and a Full Professorship in 2002. In 2000 he was a Visiting Scholar at University of South Florida, USA, and in 2004 a Visiting Scholar at Vanderbilt University, USA, each time for one semester. He also visited on several occasions Universidad de Coimbra, Portugal, for periods ranging between 3 to 5 months. During the last two years he has been Head of the Departmento de Matemáticas at Univ. Carlos III de Madrid. Recently, he was reelected for another two years term.

His mathematical interests lie in areas of rational approximation of analytic functions and orthogonal polynomials. He has published more than 80 mathematical research articles, two books, and co-edited two volumes of conference proceedings. Together with Professor Valery V. Vavilov, Guillermo wrote an excellent book on rational approximation which was to be published by a Cuban editorial. Unfortunately, because of the economical crisis in Cuba, it was never published. Moreover, he has supervised ten doctoral thesis and has twelve "mathematical descendants":

1986. René Piedra de la Torre, Row Convergence of Multipoint Padé Approximants, Universidad de la Habana.

1987. Jesús Illán González, Rational Approximation of Analytic Functions and an Associated Linear Problem, Universidad de la Habana.

1987. René Hernández Herrera, Relative Asymptotics of Orthogonal Polynomials with Respect to Varying Measures and Applications, Universidad de la Habana.

1996. Manuel Bello Hernández, Asymptotics of Orthogonal Polynomials with Respect to Varying Measures on a Semiaxis and Applications, Universidad de la Habana.

2004. Judit Mínguez Ceniceros, Extremal Polynomials and Fourier-Padé Approximants, Universidad de La Rioja.

1997. Francisco Cala Rodríguez, Convergence of Interpolating Rational Functions with Preassigned Poles, Umeå University.

1998. Héctor Pijeira Cabrera, Moment Problem and Asymptotic Properties of Sobolev Orthogonal Polynomials, Universidad Carlos III de Madrid.

2007. Ignacio Pérez Izquierdo, Asymptotics of Sobolev Extremal Polynomials, Universidad de la Habana.

1998. Abel Fernández Infante, On the Boundary Behavior of Rows of Padé Approximants, Universidad de la Habana.

2000. Angel Ribalta, Rational Approximation of Transfer Functions of Infinite Dimensional Systems, Universität Bremen.

2000. Bernardo de la Calle Ysern, Asymptotic Properties of Varying Orthogonal Polynomials and Rational Approximation, Universidad Carlos III de Madrid.

2004. Ulises Fidalgo Prieto, Hermite-Padé Approximation and Applications, Universidad Carlos III de Madrid.

Guillermo's wife, Elena, is a chatty woman always ready to invite friends to a good Cuban supper. His older daughter Dayami, made him grandpa a year ago for the first time. He also has twin sons Abey and Inti. Abey is following his father's steps as a graduate student in Math at Vanderbilt University under the supervision of Professor Edward B. Saff.

He heads an important group of researchers supported by the Science Found of the government of Spain. Guillermo is particularly active on the international scene. He has been an invited speaker to many international conferences on approximation theory and since 1996 serves on the Editorial Board of Journal of Approximation Theory.

Together with his high scientific qualities and great capacity for work, Guillermo is very approachable and a close friend. We are honored to have him as colleague, advisor, and friend.

# 2 Rational approximation and orthogonal polynomials on the real line.

These notes try to highlight the results of Guillermo in rational approximation and orthogonal polynomials on the real line. The work reviewed here shows his understanding of these subjects. He has developed the theory of orthogonal polynomials with respect to varying measure showing many applications not only for solving problems in rational approximation but also for studying questions of orthogonal polynomials involving fixed measures. We do not give all the theorems of his papers, only those most relevant. Of course, the authorship of theorems in a paper with several authors is shared. Some details in definitions and theorems are avoided in order to simplify the exposition. The review is in chronological order, although if there exists a strong relation between several papers we put them together. To save space, along this exposition the notation will be the same, so in many cases it does not agree with the one employed in the papers.

The paper [A1] announces the main results contained in [A3]. Guillermo considered multipoint Padé approximants for functions of Stieltjes type,

$$\widehat{\mu}(z) = \int_0^{+\infty} \frac{d\mu(x)}{z-x}, \tag{1}$$

where $\mu$ is a positive Borel measure on $[0, +\infty)$ with infinite points in its support[1] and finite generalized moments,

$$c_{n,k} = \int \frac{x^k}{\omega_{n,k+1}(x)} d\mu(x) < +\infty.$$

Here

$$\omega_{n,k+1}(x) = \prod_{j=1}^{k+1} (1 - \alpha_{n,j}^{-1} x), \; k = 0, \ldots, 2n-1,$$

and $\alpha = \{\alpha_{n,k}, \; k = 1, \ldots, 2n, \; n \in \mathbf{N}\}$ denotes the table of interpolation points which belong to $[-\infty, a]$, $a < 0$. Given $n \in \mathbf{N}$, the multipoint Padé approximant for $\widehat{\mu}$ associated to $\alpha$ is the unique rational function $\pi_n = p_{n-1}/q_n$ interpolating $\widehat{\mu}$ at $\{\alpha_{n,1}, \ldots, \alpha_{n,2n}\}$ (counting multiplicity), where $p_{n-1}$, $q_n$ are polynomials, $\deg(p_{n-1}) \le n-1$, $\deg(q_n) \le n$. In [A3] he proved:

**Theorem 2.1** *If*

$$\lim_n \sum_{k=1}^{2n} \frac{1}{\sqrt[2n]{c_{n,k}}} = \infty, \tag{2}$$

*then* $\lim_{n \to +\infty} \pi_n(z) = \widehat{\mu}(z)$ *uniformly on each compact subset of* $\mathbf{C} \setminus [0, +\infty)$.

The proof of this result combines techniques from analytic function theory and the moment problem. Moreover, he uses several formulas:

[1] When the support of the measure $\mu$ is finite, $\hat{\mu}$ is a rational function and all the multipoint Padé approximants for large $n$ coincide with $\hat{\mu}$.

- The denominators of the multipoint Padé approximants satisfy orthogonality relations with respect to varying measures (which change with the degree of the polynomial)

$$\int x^k q_n(x)\,\frac{d\mu(x)}{\omega_{2n}(x)} = 0, \quad k = 0, 1, \ldots, n-1, \tag{3}$$

  where $\omega_{2n}(x) = \omega_{n,2n}(x)$.

- $\pi_n(z) = \sum_{k=1}^{n} \frac{\lambda_{n,k}}{z - x_{n,k}}$, where $x_{n,k}$, $k = 1, \ldots, n$, denote the zeros of $q_n$.[2]

- An interpolatory rational quadrature formula

$$\int \frac{p(x)}{\omega_{2n}(x)} d\mu(x) \sum_{k=1}^{n} \lambda_{n,k} \frac{p(x_{n,k})}{\omega_{2n}(x_{n,k})} \tag{4}$$

  for every polynomial $p$ of degree at most $2n-1$.

Here we write explicitly (3) and (4) because in several papers Guillermo deals with these relations and reveals their own interest.

If the moments of the measure $\mu$ satisfy Carleman's condition or the support of $\mu$ is a compact set, then (2) holds for every interpolation table of the kind under consideration. Thus, Guillermo's theorem is a generalization of Markov's theorem and the Stieltjes-Carleman theorem.

In [A2] Gonchar and Guillermo describe the rate of convergence of multipoint Padé approximants when the measure has compact support, $\operatorname{supp}(\mu)$, on $\mathbf{R}$. Let $g_{\operatorname{supp}(\mu)}(z, \zeta)$ be Green's function of the complement of $\operatorname{supp}(\mu)$ with logarithmic singularity at the point $\zeta$. Let $\Delta$ denote the convex hull of the support of $\mu$ and $D = \mathbf{C} \setminus \Delta$. For each $n \in \mathbf{N}$, the interpolation points $\alpha_{n,1}, \ldots, \alpha_{n,2n}$ are symmetric with respect to $\mathbf{R}$.

**Theorem 2.2** *If the limit points $\alpha'$ of the interpolation points are contained in a compact set $E \subset D$, then for every compact set $K \subset D$ we have*

$$\limsup_{n\to\infty} \left( \max_{z\in K} |\widehat{\mu}(z) - \pi_n(z)| \right)^{1/(2n)} \le e^{-\kappa(\operatorname{supp}(\mu), K, \alpha')}$$

*where $\kappa(\operatorname{supp}(\mu), K, \alpha') = \inf\{g_{\operatorname{supp}(\mu)}(z, \zeta) : z \in K,\ \zeta \in E\}$.*

The proof of this result uses techniques of potential theory; the main idea is to apply the maximum principle to

$$\psi_n(z) = \log|\widehat{\mu}(z) - \pi_n(z)| + \sum_{k=1}^{2n} g_J(z, \alpha_{n,k}),$$

which is a subharmonic function in $\overline{\mathbf{C}} \setminus J$. Here $J$ denotes a finite union of intervals satisfying $\operatorname{supp}(\mu) \subset J \subset \Delta$.

[2] $\lambda_{n,k}$ is defined by the equality itself.

Assume that $\operatorname{supp}(\mu)$ is a finite union of intervals and $\lambda = \lambda_{\operatorname{supp}(\mu)} - \lambda_E$ denotes the equilibrium measure for the condenser $(E, \operatorname{supp}(\mu))$, i.e. $\lambda_E$ and $\lambda_{\operatorname{supp}(\mu)}$ are probability measures whose supports are included in $E$ and $\operatorname{supp}(\mu)$, respectively, and

$$\int \log \frac{1}{|z-x|}\, d\lambda(x) = \begin{cases} h_E, & z \in E, \\ h_{\operatorname{supp}(\mu)}, & z \in \operatorname{supp}(\mu), \end{cases}$$

where $h_E$ and $h_{\operatorname{supp}(\mu)}$ are constants. They also proved

**Theorem 2.3** *If $\mu' > 0$ a.e. on $\operatorname{supp}(\mu)$ and $\frac{1}{2n}\sum_{k=1}^{2n} \delta_{\alpha_{n,k}} \xrightarrow{*} \lambda_E$,*[3] *then*

1. $\lim_{n\to\infty} \left( \max_{z\in E} |\widehat{\mu}(z) - \pi_n(z)| \right)^{1/(2n)} = e^{h_{\operatorname{supp}(\mu)} - h_E} < 1.$
2. $|Q_n(z)|^{1/n} \rightrightarrows \exp\left( \int \log|z-x|\, d\lambda_{\operatorname{supp}(\mu)}(x) \right)$, $z \in \mathbf{C} \setminus \operatorname{supp}(\mu)$, *where $Q_n$ (the denominator of $\pi_n$) is the monic orthogonal polynomial satisfying (3).*

This result was extended to Markov type meromorphic functions in [A4], that is, to a Markov function plus a rational function with real coefficients. If the rational function has the form

$$r(z) = \frac{s_{d-1}(z)}{t_d(z)} = \sum_{j=1}^{l} \sum_{k=1}^{k_j} \frac{A_{j,k}}{(k-1)!(z-a_j)},$$

then the denominator $q_n$ of the multipoint Padé approximant of degree $n$ for $\widehat{\mu}(z) + r(z)$ satisfies

$$\int p(x)\, q_n(x) \frac{d\mu(x)}{\omega_{2n}(x)} + \sum_{j=1}^{l} \sum_{k=1}^{k_j} A_{j,k} \left( \frac{p(z)\, q_n(z)}{\omega_{2n}(z)} \right)^{(k-1)}_{z=a_j} = 0, \qquad (5)$$

for every polynomial $p$ of degree not greater than $n-1$. The main idea is, first, to estimate the remainder by a rational function, next, to prove convergence in capacity, and finally, to prove uniform convergence on each compact subset of the corresponding region. Moreover, in [A5] a similar theorem was proved for Stieltjes type meromorphic functions if the measure lies in $[a, \infty)$, $a > 0$, and the poles of the rational function have multiplicity one and belong to $[0, a]$, observing that each pole of the rational perturbation attracts a simple pole of the Padé approximants. If the rational function has real coefficients and the moments of the measure $\mu$ satisfy the Carleman condition, general convergence results of Padé approximants for Stieltjes type meromorphic function were proved in [A6], [A10], and [A13]. He also observes that the same results are true for Padé approximants "near the main diagonal." Guillermo's sharpest result on the convergence of Padé approximants for Stieltjes type meromorphic functions, when the rational function has real coefficients, is contained in [A14]. He proves the following theorem.

[3] When this condition holds, they say the interpolation points have extremal distribution. Such interpolation tables are very important in approximation theory. $\delta_x$ denotes the probability measure with support at $\{x\}$.

**Theorem 2.4** *Let $f = \widehat{\mu} + r$, where $r = s_d/t_d$ is an arbitrary rational function with real coefficients and poles in $D$ ($\deg(t_d) \leq d$, $r(\infty) = 0$). If for each polynomial $l$ positive on $[0, \infty)$ such that $\deg(l) \leq d$ the moment problem for the sequence $c_j^* = \int x^j x^m t_d(x)\, l(x) d\mu(x)$, $j \in \mathbf{N}$, is determinate, then the sequence $\{\pi_{n+m,n}\}, n \in \mathbf{N}$ satisfies:*

**i)** *For all $n$ sufficient large the number of poles of $\pi_{n+m,n}$ in $D$ is equal to the number of poles of $r$; the poles of $\pi_{n+m,n}$ in $D$ tend as $n \to \infty$ to the poles of $r$ in such a way that each pole of $r$ attracts as many poles of $\pi_{n+m,n}$ as its multiplicity.*

**ii)** *The sequence $\{\pi_{n+m,n}\}, n \in \mathbf{N}$, converges uniformly to $f$ on every compact set contained in $D \setminus \{r(z) = \infty\}$.*

Guillermo proves the convergence of multipoint Padé approximants for Stieltjes type meromorphic functions (the rational function has real coefficients) in [A7]. The measure and the interpolation points satisfy the equilibrium condition given by the generalized Carleman condition (2).

Gonchar posed a conjecture about the rows of Padé approximants: *"Let $\{q_{n,m} : n \in \mathbf{N}\}$ be the denominators of the $m-$th row of the Padé approximants of an analytic function $f$ at the origin. If it is known that there exists a polynomial $l_m(z) = \prod_{j=1}^{m}(z - a_j)$ such that $\lim_n q_{n,m}(z) = l_m(z)$ and $0 < |a_1| \leq \ldots \leq |a_{k-1}| < |a_k| = \ldots = |a_m| = \rho$, he asks if $\rho$ is the radius of the largest open disk with center at the origin to which $f$ extends as a meromorphic function having no more than $k-1$ poles counting their multiplicity ($k-1-$meromorphic radius)."*[4] The paper [A9] contains a solution of this problem for $k = m$. This an important result for applications of Padé approximant. The proof of this theorem follows a careful analysis of many formulas for the rows of Padé approximants, in particular, it uses the Fabri quotient formula (the aforementioned conjecture for $m = 1$.) As a by-product it is also obtained a formula for the $m-$meromorphic radius.

The papers [A11] and [A23] contain an interesting point of view on the convergence of multipoint Padé approximants. It is proved an equivalence between the convergence of generalized Gaussian quadrature (numerical integration formulas satisfying (4)) and the convergence of multipoint Padé approximants for Stieltjes functions.

In [A12] a result of overconvergence for rational approximants of meromorphic functions is proved. The authors compare rational interpolants at the origin with some fixed poles at infinity and rational interpolants along a table of points of $\{z : z^n = R\}$, $R > 0$. This result extends one of Saff, Sharma, and Varga. The papers [A18] and [A38] include extensions of the classical theorems of Abel, Tauber, and Fatou to meromorphic functions in a disk with a finite number of poles. As an example, we include a rational version of Abel's theorem.

**Theorem 2.5** *If $f$ is an analytic function at the origin which has a meromorphic extension to $\{z : |z| < 1\}$ with exactly $m$ poles and $\lim_n \pi_{n,m}(1)$ exists, then $\lim_{x\to 1^-} f(x) = \lim_n \pi_{n,m}(1)$.*

[4] If it gets information about the function from the behavior of Padé approximant, then it is called an inverse result in the theory of Padé approximants.

This paper supports a principle announced by Guillermo: *Every theorem for Taylor's series has an analogous for Padé row.*

The paper [A15] mainly surveys results concerning the inverse problem for rows of Padé approximants. But it also reviews results on the diagonal and multipoint Padé approximants. Some open problems were presented.

A new perspective for research was presented in [A17] regarding relative asymptotics of orthogonal polynomials with respect to varying measures. It plays an important role in the study of multipoint Padé approximation of meromorphic functions as well as in the solution of questions on orthogonal polynomials involving fixed measures.

We strongly recommend [A20] because Guillermo shows there how some of his results in multipoint Padé approximants for meromorphic functions were developing.

In [A25] Guillermo explicitly studied for the first time orthogonal polynomials with respect to varying measures on the unit circle. His aim was to establish the convergence of multipoint Padé approximants of Markov type meromorphic functions, but he needed the ratio asymptotics of orthogonal polynomials with respect to varying measures on the real line. Guillermo proved:

**Theorem 2.6** *If $\mu' > 0$ a.e. on $(-1,1)$, $\operatorname{supp}(\mu) \subset [-1,1]$, then, for every integer $j$,*

$$\lim_{n\to\infty} \frac{Q_{n,n+j+1}(z)}{Q_{n,n+j}(z)} = \frac{1}{2}\left(z + \sqrt{z^2-1}\right), \quad z \in \mathbf{C} \setminus [-1,1], \tag{6}$$

*where the convergence is uniform on compact sets of $\mathbf{C} \setminus [-1,1]$ and $Q_{n,m}$ is the $m$-th monic orthogonal polynomial with respect to $\frac{d\mu(x)}{\omega_{2n}(x)}$.*

This result generalizes a theorem of Rakhmanov for orthogonal polynomials with respect to a fixed measure which is one of the most important contributions to the subject of orthogonal polynomials at the end of last century. Máté, Nevai, and Totik gave a simpler proof of the Rakhmanov theorem. Guillermo's proof of (10) follows their ideas. Let $\rho$ be a positive Borel measure on $\Gamma = \{z \in \mathbf{C} : |z| = 1\}$. The derivative of the measure $\rho$ with respect to the Lebesgue measure is denoted by $\rho'$. He fixes $r_0 < 1$ and a sequence of monic polynomials $(W_n(z))$ whose zeros are in the disk $\{z \in \mathbf{C} : |z| \le r_0\}$. For each $n \in \mathbf{N}$, he defines the measure

$$d\rho_n(z) = \frac{d\rho(z)}{|W_n(z)|^2}, \quad z \in \Gamma \tag{7}$$

and considers the polynomials $\varphi_{n,m}(z) = \alpha_{n,m} z^m + \cdots, \alpha_{n,m} > 0$, which are orthogonal on $\Gamma$ with respect to the measure $\rho_n$, i.e.

$$\int_\Gamma \overline{z}^j \varphi_{n,m}(z) d\rho_n(z) = 0, \quad j = 0, 1, \ldots, m-1, \tag{8}$$

$$\frac{1}{2\pi} \int_\Gamma \left|\varphi_{n,m(z)}\right|^2 d\rho_n(z) = 1. \tag{9}$$

Guillermo proved

**Theorem 2.7** *If $\rho' > 0$ almost everywhere on $\Gamma$, then for every integer $k$ the following relation holds*

$$\lim_{n\to\infty} \frac{\varphi_{n,n+k+1}(z)}{\varphi_{n,n+k}(z)} = z, \quad |z| \geq 1 \tag{10}$$

*and the convergence is uniform on the indicated region.*

For the proof of this theorem Guillermo obtains a result of independent interest which shows an equilibrium relation between $W_n$ and $\varphi_{n,n+k}$.

**Theorem 2.8** *If the assumptions of the above theorem hold, then for every integer $k$, the following relation holds:*

$$\frac{|W_n(z)|^2}{|\varphi_{n,n+k}(z)|^2}|dz| \xrightarrow{*} d\rho. \tag{11}$$

Using Theorem 2.6 Guillermo deduces the comparative asymptotics for the monic denominators, $Q_n$, of the multipoint Padé approximants of Markov type meromorphic functions where the rational function $r$ has complex coefficients. An auxiliary, but important result by itself, was the following theorem:

**Theorem 2.9** *If $\mu' > 0$ almost everywhere in $(-1,1)$, then*

$$\lim_{n\to\infty} \frac{Q_n(z)}{Q_{n,n}(z)} \frac{\prod_{j=1}^d (\varphi(z) - \varphi(a_j))^2}{2^d \varphi^d(z) \prod_{j=1}^d (z - a_j)}, \tag{12}$$

*where $\varphi(z) = z + \sqrt{z^2 - 1}$ and $\prod_{j=1}^d (z - a_j)$ is the denominator of $r(z)$.*

With this result he proved an extension of Theorem 2.2 for Markov type meromorphic functions $\widehat{\mu} + r$.

**Theorem 2.10** *If the limit points $\alpha'$ of the interpolation points are contained in a compact set $E \subset D' = D \setminus \{r = +\infty\}$, and $\mu' > 0$ a.e. on $(-1,1)$, then*

1. *Each pole of $r$ attracts, when $n \to \infty$, as many poles of $\pi_n$ according to its multiplicity; the rest of poles of $\pi_n$ approach to $(-1,1)$.*
2. *For every compact set $K \subset D$ we have*

$$\limsup_{n\to\infty} \left( \max_{z\in K} |\widehat{\mu}(z) + r(z) - \pi_n(z)| \right)^{1/(2n)} \leq e^{-\kappa(\mathrm{supp}(\mu), K, \alpha')}$$

*where $\kappa(\mathrm{supp}(\mu), K, \alpha') = \inf\{g_{\mathrm{supp}(\mu)}(z,\zeta) : z \in K,\, \zeta \in E\}$.*

After this paper, Guillermo published several papers where asymptotic properties of orthogonal polynomials with respect to varying measures were established. In [A31] he extended (6), (10), and (11). The assumptions on the "interpolation points" (the zeros of $W_n$ or $\omega_{2n}$) were weakened allowing them to approach the support of the measure; for example, for the unit circle[5]:

[5] If this condition holds he says that $(\rho, W_n, k)$ is admissible

1. $\rho' > 0$ a.e. on $\Gamma$.
2. If $k < 0$, $\sup_n \int \prod_{j=1}^{-k} |z - w_{n,j}|^{-1} \, d\rho(z) \le M < +\infty$.
3. $\displaystyle\lim_{n\to\infty} \sum_{j=1}^{n} (1 - |w_{n,j}|) = +\infty$.

where $w_{n,j}, j = 1, \ldots, n$ are the zeros of $W_n$. Under this condition, he also proved several strong and weak asymptotic results; for instance,

$$\lim_n \int \left( \left| \frac{\varphi_{n,n+k}(z)}{W_n(z)} \right| (\rho'(z))^{1/2} - 1 \right)^2 |dz| = 0,$$

$$\lim_n \int f(z) \frac{\varphi_{n,n+k}(z)\overline{\varphi_{n,n+k+1}(z)} z^l}{|W_n(z)|^2} d\rho(z) = \int f(z) \, |dz|,$$

for every bounded Borel-measurable function $f$ on $\Gamma$, where $|dz|$ stands for the normalized Lebesgue measure on the unit circle. Moreover, he also obtained in this paper relative asymptotics for varying orthogonal polynomials whose weights are varying rational functions with complex coefficients which do not vanish on the support of the measure and have fixed degree.

Guillermo goes on with orthogonal polynomials with respect to varying measures. In [A21], [A24], [A27], and [A28] studied other problems like strong and relative asymptotics for orthogonal polynomials with respect to varying measures on the unit circle and on intervals of the real axis. With all these results at hand, in [A32] he proved a result on relative asymptotics for orthogonal polynomials with respect to a fixed measure on $\mathbf{R}$. The main idea is to reduce the problem to varying orthogonal polynomial on the unit circle. Guillermo showed that orthogonal polynomials with respect to varying measures are an effective tool not only for solving problems on multipoint rational approximation but also for studying questions on orthogonal polynomials involving fixed measures and observed that orthogonal polynomials with respect to varying measures on the unit circle provide a unified approach to the study of orthogonal polynomials on finite and infinite intervals.

Let $\mu$ be a finite positive Borel measure on $\mathbf{R}$ with finite moments. Let $g$ be a nonnegative function on $\mathbf{R}$, $g \in L^1(\mu)$, and set $\mathcal{H}_n(g\,d\mu, z) = \frac{h_n(g\,d\mu,z)}{h_n(g\,d\mu,i)}$ and $\mathcal{H}_n(\mu, z) = \frac{h_n(\mu,z)}{h_n(\mu,i)}$, where $h_n(g\,d\mu, z)$ and $h_n(\mu, z)$ denote the orthonormal polynomials of degree $n$ with respect to the measures $g\,d\mu$ and $\mu$, respectively. Guillermo proved:

**Theorem 2.11** *Assume that $\mu' > 0$ a.e. on $\mathbf{R}$ and the moments of $\mu$ satisfy the Carleman condition. If there exists an algebraic polynomial $Q$ and $p \in \mathbf{N}$ such that $|Q|g^{\pm}/(1 + x^2)^p$ belong to $L^\infty(\mu)$, then*

$$\lim_{n\to\infty} \frac{\mathcal{H}_n(g\,d\mu, z)}{\mathcal{H}_n(\mu, z)} \frac{S(\mu, G, z)}{S(g, G, i)}, \quad z \in G,$$

*where $G = \{z \in \mathbf{C} : \Im z > 0\}$, and $S(g, G, z)$ denotes the Szegő function for $g$ with respect to the region $G$, i.e.*

$$S(g, G, z) = \exp\left\{ \frac{1}{2\pi i} \int_{\mathbf{R}} \log g(x) \frac{xz + 1}{z - x} \frac{dx}{1 + x^2} \right\}$$

One year later Guillermo published another striking paper, [A30]. There he solved a conjecture posed by A.A. Gonchar 10 years earlier about the convergence of Padé approximants of Stieltjes type meromorphic functions. Gonchar proved the convergence of Padé approximants to Markov-type meromorphic function whose measure $\mu$ is supported on an interval of the real line, and satisfies $\mu' > 0$; moreover the rational perturbation has complex coefficients. Later on, Rakhmanov showed that the convergence does not hold for arbitrary positive measure $\mu$. Guillermo gave a very general sufficient condition to get convergence of Padé approximants for Stieltjes-type meromorphic functions whose rational perturbation has complex coefficients. Let $\mu$ be a positive Borel measure on $[0, +\infty)$ and $r$ a rational function with complex coefficients whose poles, $\{a_1, \ldots, a_d\}$ lie in $\mathbf{C} \setminus [0, \infty)$ and $r(\infty) = 0$. Let $\pi_n = P_{n-1}/Q_n$ be the Padé approximant of order $(n-1, n)$ of $\widehat{\mu} + r$, where $Q_n$ is normalized so that $Q_n(1) = (-1)^n$ (since Padé approximants are invariant with respect to linear change of variables, with this condition no generality is lost). Let $\mathcal{L}_n$ denote the orthogonal polynomials of degree $n$ with respect to $\mu$ normalized by the condition $\mathcal{L}_n(-1) = (-1)^n$. The main theorem in this paper states:

**Theorem 2.12** *If $\mu' > 0$ on $(0, \infty)$ and their moments satisfy the Carleman condition, then*

1.
$$\lim_{n\to\infty} \frac{Q_n(z)}{\mathcal{L}_n(z)} = \prod_{j=1}^{d} \frac{(1+z)(1+a_j)(\Phi(z) - \Phi(a_j))}{4\Phi(z)(z - a_j)},$$
*where $\Phi(z) = \frac{\sqrt{z}+i}{\sqrt{z}-i}$, and the convergence is uniform on each compact subset of $\mathbf{C} \setminus \{[0, \infty) \cup \{a_1, \ldots, a_d\}\}$.*

2. *The sequence $\{\pi_n\}$ converges to $\widehat{\mu} + r$, uniformly on each compact subset of $\mathbf{C} \setminus \{[0, \infty) \cup \{a_1, \ldots, a_d\}\}$.*

Roughly speaking the line of the proof of this result is to transform $(0, \infty)$ into $(-1, 1)$ with a linear fractional function; here the interpolation at $\infty$ changes to interpolation at $1$. The denominator of the Padé approximants interpolating the associated Markov type meromorphic function at $1$ is orthogonal with respect to a varying measure on $(-1, 1)$. The cornerstone is to obtain relative and ratio asymptotics of these new orthogonal polynomials. Finally Guillermo translates this to an analogous problem for orthogonal polynomials on the unit circle and solves this new challenge.

The papers [A26] and [A29] present an update review of the results on rational approximation of analytic functions underlining its interplay with the theory of orthogonal polynomials and potential theory.

In [A43] the rate of convergence of two point-Padé approximants for Stieltjes type meromorphic functions is studied. The interpolating conditions are proportionally distributed between $0$ and $\infty$ The measure for the Stieltjes function is $d\mu(x) = x^\alpha e^{-\tau(x)} dx$, $x \in (0, \infty)$, where $\alpha \in \mathbf{R}$, and $\tau(x)$ is a continuous function on $(0, +\infty)$ that satisfies, for some $\gamma > 1/2$ and $s > 0$, $\lim_{x\to 0^+} (sx)^\gamma \tau(x) = \lim_{x\to+\infty} (sx)^{-\gamma} \tau(x) = 1$. The main auxiliary result is the logarithmic asymptotics of Laurent-type orthonormal polynomials $h_n$ for $\frac{d\mu(x)}{x^{\lambda_n}}$, $0 \le \lambda_n \le 2n$.

**Theorem 2.13** *Assume that* $\lim \frac{\lambda_n}{2n} = \theta \in [0,1]$. *Uniformly on compact sets of* $\mathbf{C} \setminus [0, +\infty)$, *we have*

$$\lim_{n\to\infty} \frac{\log \left| h_n(z) z^{-\lambda_n/2} \right|}{(2n)^{1-1/(2\gamma)}} = D(\gamma) \left\{ (1-\theta)^{1-1/(2\gamma)} \Im(\sqrt{sz}) + \theta^{1-1/(2\gamma)} \Im(\sqrt{\frac{1}{sz}}) \right\},$$

*where* $D(\gamma) = \frac{2\gamma}{2\gamma-1} \left( \frac{\Gamma(\gamma+\frac{1}{2})}{\sqrt{\pi}\Gamma(\gamma)} \right)$ *and* $\Gamma(z)$ *is Euler's gamma function.*

Guillermo proved in [A59] and [A60] convergence for rational interpolants with prescribed poles and generalized rational interpolants with partially prescribed poles of analytic functions on their natural region of analyticity. Some applications to dynamical systems are shown. Let $a > 0$, $\Gamma = \{z \in \mathbf{C} : \Re(z) = -a\}$ and $f$ denote an holomorphic function in a region $G \supset \{z \in \mathbf{C} : \Im(z) \geq -a\}$. Set $\Gamma_R = \Gamma \cap \{|z| \leq R\}$. It is assumed that there exist $C > 0, \alpha > 0$, and $R > 0$ such that for all $z$ with $\Im(z) > -a$ and $|z| > R$ we have

$$|f(z)| \leq \frac{C}{|z|^\alpha},$$

and the limit in the right-hand side of the following definition exists.

$$\int_\Gamma f(z)\, dz \stackrel{def}{=} \lim_{R\to+\infty} \int_{\Gamma_R} f(z)\, dz.$$

**Theorem 2.14** *Let* $\{T_n : n \in \mathbf{N}\}$ *be a sequence of polynomials with no zeros in* $\{\Re z \geq -a\}$ *and* $\deg(T_n) = n$. *Let the points* $\{\alpha_{n,k} : k = 1, \ldots, n; n \in \mathbf{N}\}$ *lie in* $\{\Re z \geq 0\}$. *Set* $\omega_n(z) = \prod_{k=1}^n (z - \alpha_{n,k})$. *Assume that the sequence*

$$A_n(z,t) = \frac{\omega_n(z) T_n(t)}{\omega_n(t) T_n(z)},\ n \in \mathbf{N},$$

*converges uniformly to zero on each bounded subset of* $\Gamma \times i\mathbf{R}$, *and is uniformly bounded on this set. Let* $r_n = S_n/T_n$ *be the rational function (with fixed poles) of degree* $n$ *which interpolates* $f$ *at* $\{\alpha_{n,k} : k = 1, \ldots, n\}$. *Then* $\{r_n\}$ *converges uniformly to* $f$ *on* $\{z : \Im z \geq 0\}$.

This theorem allows the authors to approximate by rational functions all stable time-delay systems with transfer functions of the form

$$f(z) = \frac{\sum_{j=1}^m q_j(z) e^{-\alpha_j z}}{p_0(z) + \sum_{j=1}^m p_j(z) e^{-\gamma_j z}},$$

where $p_j$, $q_j$ are polynomials with $\deg(p_0) > \deg(p_j)$, $\deg(p_0) > \deg(q_l)$, for all $j \neq 0$ and all $l$, and $\gamma_j \geq 0$, $\alpha_j \geq 0$. For these types of functions particular interpolation tables and fixed poles were exhibited. Other class of transfer functions are also considered and convergence theorems for rational interpolation with fixed poles are given.

Guillermo's study of interpolating rational approximants, whose poles are partially fixed and approximate Markov functions, was displayed in [A50] and [A62]. Let $\mu$ be a finite positive Borel measure whose support is contained in $B=[-1,1]$. Let $\{B_n\}_{n\in\mathbf{N}}$ be a sequence of monic polynomials with constant sign in $B$. Assume that $\deg(B_n)=m(n)\le n$ and

$$B_n(z)=\prod_{j=1}^{m(n)}(z-\beta_{n,j}),\quad \beta_{n,j}\in B.$$

Let $A\subset\mathbf{C}\setminus B$ be a compact set that is symmetric with respect to the real line and the interpolation points $\alpha_{n,j}: j=1,\ldots,2n-m(n)$, contained in $A$, are also symmetric with respect to the real line, $n\in\mathbf{N}$ . Set

$$\omega_n(z)=\prod_{j=1}^{2n-m(n)}(z-\alpha_{n,j}),\quad n\in\mathbf{N}.$$

Let $\pi_n=P_n(z)/(B_n\,Q_{n-m(n)})$ be the rational function which interpolates the function $\widehat{\mu}$ at $\alpha_{n,j}$ : $j=1,\ldots,2n-m(n)$. Assume that there exists

$$\lim_n\frac{m(n)}{n}=\theta\in[0,1]$$

and the measures $\frac{1}{n}\sum_{j=1}^{2n-m(n)}\delta_{\alpha_{n,j}}$ and $\frac{1}{n}\sum_{j=1}^{m(n)}\delta_{\beta_{n,j}}$ converge in the weak-star topology to $(2-\theta)\lambda$ and $\theta\beta$, respectively, where $\operatorname{supp}(\lambda)\subset A$ and $\operatorname{supp}(\beta)\subset B$. Moreover, let $\lambda^*$ denote the balayage of $\lambda$ onto $B$. Finally, it is assumed that $(2-\theta)\lambda^*-\theta\beta\ge 0$. Thus, we get

**Theorem 2.15** *If $\mu'>0$ a.e. on $(-1,1)$, then on each compact set $K\subset\mathbf{C}\setminus(A\cup B)$, it holds*

$$\limsup_n\sup_{z\in K}|\widehat{\mu}(z)-\pi_n(z)|^{1/n}=\sup_{z\in K}\exp\{(\theta-2)(\omega^*-V_{\lambda^*-\alpha}(z)\},$$

*where $\omega^*=\int g_B(t,\infty)\,d\alpha(t)$, $g_B(t,\infty)$ denotes the Green function for $\overline{\mathbf{C}}\setminus B$ with singularity at $\infty$, and $V_{\lambda^*-\alpha}(z)$ is the logarithmic potential of the measure $\lambda^*-\alpha$.*

In [A65] the same problem was studied for Markov-type meromorphic functions; this paper contains a wide use of potential theory in rational approximation. A similar problem was studied in [A51] for Padé-type approximants (Padé approximants which some or all poles fixed) of Stieltjes functions whose weights are $d\rho_\gamma(x)=\exp\{-f_\gamma(x)\}\,dx$, $x\in\mathbf{R}$, where $\lim_{x\to\infty}f_\gamma(x)|x|^{-\gamma}=1$, and $\gamma>1$. The fixed poles of the approximants are zeros or orthogonal polynomials of degree $m(n)$ with respect to the weight $d\rho_\beta(x)$. If $1<\beta<\gamma$, the rate of convergence of the rational approximants to the Stieltjes transform was described.

In all these papers on Padé-type approximants the rational interpolants have fixed poles whose multiplicities are even or the polynomials with zeros at the poles have constant sign on the support of the measure which defines the Markov function. In [A66] and [A71] Padé-type approximants whose fixed poles are simple and lie in the support of the measure associated to the Markov function were studied. The main idea was to obtain the asymptotic behavior for the Stieltjes polynomials because the free poles of the approximants are the zeros of these polynomials while the fixed poles are the zeros of orthogonal polynomials. Let $\mu$ be a finite, positive Borel measure on $\mathbf{R}$ whose compact support contains infinitely many points. Set $\Omega = \overline{\mathbf{C}} \setminus \operatorname{conv}(\operatorname{supp}(\mu))$. Let $\{p_n\}_{n\in\mathbf{N}}$ be the sequence of orthonormal polynomials with respect to $\mu$, whose leading coefficient $\kappa_n$ is positive. This sequence is said to have regular asymptotics and we write $\mu \in \mathbf{Reg}$, if

$$\lim_n |p_n(z)|^{1/n} = \exp\{g_\Omega(z,\infty)\},$$

uniformly on each compact subset of $\Omega$.

The $n-$th Stieltjes polynomial is the unique monic polynomial $S_n$ of degree $n$ which satisfies

$$\int x^j S_n(x)\, p_{n-1}(x)\, d\mu(x) = 0, \quad j = 1, \ldots, n-1.$$

**Theorem 2.16** *If $\mu \in \mathbf{Reg}$ and $\operatorname{Cap}(\operatorname{supp}(\mu)) > 0$, then*

$$\lim_n \left| \frac{S_{n+1}(z)}{\kappa_n} \right|^{1/n} = \exp\{g_\Omega(z,\infty)\},$$

*uniformly on each compact subset of $\Omega$; and the set of accumulation points of the zeros of $\{S_n\}_{n\in\mathbf{N}}$ is contained in $\operatorname{supp}(\mu)$.*

# 3 Sobolev orthogonality.

It is very well known that orthogonal polynomials associated with a nontrivial probability measure supported on the real line satisfy a three-term recurrence relation (second order linear difference equation) translating the fact that the multiplication operator by $x$ is a symmetric operator with respect to the standard inner product defined by such a measure.

Let $\{\mu_k\}_{k=0}^m$ be a vector of positive Borel measures supported on the real line. In the linear space $\mathbf{P}$ of polynomials with real coefficients we introduce the following Sobolev inner product

$$\langle p, q\rangle_S := \sum_{k=0}^{m} \int_{\mathbf{R}} p^{(k)}(x)\, q^{(k)}(x)\, d\mu_k(x). \tag{13}$$

In general, the sequence of polynomials orthogonal with respect to the above inner product does not satisfy a recurrence relation involving a fixed number of terms, independently of the degree of the polynomials. Thus, we loose a central tool in the analysis of our polynomials and new approaches are required.

In [26] the authors proved that there exists a polynomial multiplication operator $H$, symmetric with respect to (13), if and only if $\{\mu_k\}_{k=1}^m$ are discrete measures supported in subsets of the real line, associated with the zeros of the polynomial $H$. In such a case, taking into account the behavior of the coefficients in the corresponding higher order difference equation, then relative asymptotic properties for such polynomials in terms of the polynomials orthogonal with respect to $\mu_0$ when this measure belongs to the Nevai class $M(0,1)$ an important example of bounded measures including the Jacobi weight probability measure, among others. This problem was studied in [78] (for $m = 1$)

and [A42] (for $m$ greater than 1). This was the first contribution by Guillermo on Sobolev orthogonal polynomials. It is very remarkable that the relative asymptotics does not depend on the masses but of the localization of the mass points. Furthermore, in an unpublished manuscript by Van Assche (see [78]) the recovering of the masses on the Sobolev part of the inner product is deduced.

The location of the zeros of such orthogonal polynomials was the subject of [A46]. Here is proved that the number of zeros of the $n$th orthogonal polynomial in the interior of the support of the measure $\mu_0$ does not depend of the order of the higher derivative but of the number $p$ of positive masses involved in the inner product. At least the $n$th orthogonal polynomial has n-p changes of sign in the interior of the convex hull of the support of the measure and the amount of intervals constituted by two consecutive zeros of such a polynomial containing at least one zero of the polynomial of degree n+1 is deduced.

A more interesting situation appears when the components of the vector of measures are non-trivial probability measures supported on either bounded or unbounded subsets of the real line.

In the first case, the connection between the zeros of Sobolev orthogonal polynomials and the support of the measure plays a central role. In [A56], Guillermo introduced the concept of sequentially dominated vector measure. This yields the boundedness of the multiplication operator and, as a consequence, the zeros of Sobolev orthogonal polynomials are bounded. This allows the authors to analyze the $n$th root asymptotics of such polynomials in a very elegant way using techniques of potential theory.

On the other hand, a moment problem for Sobolev inner products is considered in [A58]. Given an infinite matrix $(s_{k,j})$ with real numbers as entries, necessary and sufficient conditions for the existence of a Sobolev inner product like (13) such that $s_{k,j} = \langle x^k, x^j \rangle_S$ are deduced.

An extension of such methods to a wide class of measures supported on compact subsets of the complex plane is analyzed in [A61]. Again, the zeros of Sobolev orthogonal polynomials are contained in a compact subset of the complex plane if the sequentially dominated property for the vector of measures is assumed. If the multiplication operator is bounded and the Sobolev product is $l$-regular, then the $n$th root asymptotics for the derivatives of order greater than $l$ is deduced.

When the vector of measures $\{\mu_k\}_{k=0}^m$ is supported on either an arc or a closed rectifiable Jordan curve in the complex plane and we add a discrete non diagonal component case involving the derivatives, in [A85] the strong asymptotic behavior for the corresponding sequences of Sobolev orthogonal polynomials using techniques of classical function theory is obtained.

In the second case, very few results (see [34]) have been done concerning the asymptotic properties of the corresponding sequences of Sobolev orthogonal polynomials, mainly when $m = 1$. In such a contribution, the authors deal with exponential weights and they deduce outer and inner strong asymptotics for the Sobolev orthogonal polynomials and their first derivative in terms of the polynomials orthogonal with respect to the measure $\mu_0$. A more general situation with higher order derivatives has been analyzed in [A79] where the contracted zero distribution and the logarithmic asymptotics of the rescaled polynomials for a wide class of measures controlled by Freud weights is deduced.

# 4 Orthogonal polynomials on the unit circle.

The theory of orthogonal polynomials on the unit circle $\Gamma := \{z : |z| = 1\}$ has different applications in signal processing, stochastic processes or, in general, in questions related with unitary operators. However, one of the most important reasons for the development of this theory is its relation with the study of orthogonal polynomials on the real line. In the beginning, this fact was used by G. Szegö in his work, who through the change of variables $z \mapsto \frac{1}{2}(z + z^{-1})$ translated the results obtained on the unit circle to polynomials on the real line.

On the other hand, the study of sequences of polynomials orthogonal with respect to varying measures on the unit circle has important consequences in approximation theory and, in particular, in rational approximation. In fact, these sequences arise in the analysis of convergence of rational interpolants for a given analytic function. Moreover, varying measures give the key to another connection between orthogonality on the real line and orthogonality on the unit circle; the conformal mapping

$$z(\xi) = \frac{\xi + i}{\xi - i}$$

transforms the real line on the unit circle $\Gamma$. (This fact can be interpreted, also, in terms of Operator Theory, using the *Cayley transform* $U_H = (H + iI)(H - iI)^{-1}$. If $H$ is a self-adjoint operator, then $U_H$ is a unitary operator. From this, we can analyze the relation between the spectral measures associated with $H$ and $U_H$, and also between their spectra.) As a consequence, for a fixed measure $d\mu$ on the real line we obtain a sequence of measures

$$d\rho_n(z) = \frac{d\rho(z)}{|z-1|^{2n}}, \quad n \in \mathbb{N},$$

supported on the unit circle, where $d\rho(z) = d\mu\,(i(z+1)/(z-1))$. We have the sequence of polynomials $\{P_n\}_{n\in\mathbf{N}}$ orthogonal with respect to $\mu$ and, for each measure $d\rho_n$, we have the corresponding sequence of orthogonal polynomials $\{\varphi_{n,m}\}_{m\in\mathbb{N}}$. These questions were studied and can be consulted in [A32, Lemma 9, pg. 526].

Thus, varying measures on the unit circle become a very important tool in the study of questions related with orthogonality on the real line. This is the basis of the methods used in several papers as, for instance, [A30], [A32], [A48], and [A76], where a problem on the real line is reduced to an analogous problem on the unit circle associated with varying measures.

In many papers, given a positive Borel measure $d\rho$, the varying measures $d\rho_n = d\rho/|W_n|^2$ were used, where $\{W_n(z)\}_{n\in\mathbf{N}}$ is a sequence of monic polynomials whose zeros are, in general, in the circle $\{|z| \le 1\}$. For each one of these measures, under obvious restrictions, it is possible to define a unique sequence $\{\varphi_{n,m}\}_{m\in\mathbf{N}}$ of polynomials with positive leading coefficient satisfying

$$\int_\Gamma \bar{z}^j \varphi_{n,m}(z) d\rho_n(z) = 0\,, \quad j = 0, 1, \ldots, m-1\,, \quad \int_\Gamma |\varphi_{n,m}(z)|^2\, d\rho_n(z) = 1, \quad .$$

In the sequel, we will use this notation.

In [A25] and [A30], extensions of the well-known Rakhmanov Theorem ([66]-[67]) on ratio asymptotics were made for varying measures on the unit circle. In [A25] it was assumed that the zeros of each $W_n(z)$ were in the closed circle $\{|z| \le r\}$ for some fixed $r < 1$ (independent of $n$). Moreover, it was assumed that $\rho' > 0$ a.e. on $\Gamma$ (where $\rho'$ denotes the derivative of the measure with respect to the Lebesgue measure). Under these conditions, the asymptotic behavior of the ratio $\varphi_{n,n+k+1}(z)/\varphi_{n,n+k}(z), n \to \infty$, in the region $\{|z| > 1\}$ was established in [A25, Theorem 2, pg. 209]. The key ingredient for the proof of this result was the weak convergence of the sequence of measures

$$\frac{|W_n|^2}{|\varphi_{n,n+k}|^2} d\theta \xrightarrow{*} d\rho, \quad n \to \infty \,, \tag{14}$$

where $d\theta$ denotes the Lebesgue measure on $\Gamma$. Under the assumptions made in this paper, this kind of convergence was proved in [A25, Lemma 1, pg. 209].

In [A30], the ratio asymptotics obtained in [A25] was studied from another perspective. In this sense, the restriction imposed in [A25] on the location of the zeros of $W_n$ was replaced in [A30] by a weaker Carleman type condition. This, so called "C-condition", was described in [A30, pgs. 216-217] using generalized moments

$$c_{n,m} := \int \frac{d\rho}{|W_{n,m}|^2} \,, \quad W_{n,m}(z) = \prod_{i=1}^{m} (z - w_{n,i}) \,, \quad m = 1, 2, \ldots, n \,,$$

where $0 \le w_{n,1} \le \cdots \le w_{n,n} \le 1$ are the zeros of $W_n$.

In [A27], the weak convergence (14) was used to determine the strong asymptotic behavior of orthogonal polynomials with respect to varying measures. In [A27, Theorem 2, pg. 257] an extension of Szegö's theorem for these polynomials was given which contains as a limit case the classical result (for a fixed measure, see [A27, Corollary 1, pg. 259]). The condition on the zeros of $W_n$ used in [A25] was substituted in [A27] by the C-condition for each $(\rho, \{W_n\}, k)$.

The concept of "admissibility" was introduced in [A24]. This concept was used in [A24] and subsequent papers, where the sufficient condition on the location of the zeros of $W_n$ used in [A25] was replaced by the weaker condition that $(\rho, \{W_n\}, k)$ be admissible for each $k \in \mathbb{Z}$.

In [A31, Theorem 3, pg. 201] an improvement of the results obtained in [A25] was given. In general, if we take $W_n(z) = z^n$ then the varying measures are $d\rho_n \equiv d\rho$. Therefore, all the results obtained in the unit circle for varying measures can be applied to this case, where we have, for any $k \in \mathbb{Z}$, that $\{\rho, \{W_n\}, k\}$ is admissible on $(0, 2\pi)$ and, also, satisfies the C-condition. In particular, the above mentioned result of E. A. Rakhmanov, proved in [66]-[67], was extended in [A31, Theorem 3, pg. 201].

In [A39, Theorem 2.7] and [A39, Theorem 2.8], more asymptotic relations were determined. In this way, the corresponding results of [57] for the case of a fixed measure were extended to varying measures. In [A39] it was assumed that $(\sigma, \{W_n\}, k)$ is admissible on $[0, 2\pi]$ for each $k \in \mathbb{Z}$. The methods used in the proofs were similar to the ones used in [A31]. A relevant fact is that Theorem 4.1 in [A39] allows us to recover the absolutely continuous part $\rho' d\theta$ and the singular part $\rho - \rho' d\theta$ of the measure $d\rho$ from the sequence $\{W_n/\varphi_{n,n+k}\}$.

Let $d\mu$ be a finite positive Borel measure with support $S \subset \mathbb{C}$. Assume that we have another measure $d\rho$ supported also on $S$, and let $\{\varphi_n(z;\mu)\}$, $\{\varphi_n(z;\rho)\}$ be the corresponding sequences of orthogonal polynomials. The aim of the study of the relative asymptotics

$$\lim_{n\to\infty} \frac{\varphi_n(z;\mu)}{\varphi_n(z;\rho)}, \quad z \in \Omega,$$

is to compare the behavior of a family of polynomials, in the region $\Omega$, in terms of the other one. Certain properties can be obtained when $d\mu = g d\rho$. In [A24], [A28], [A32], [A64], and [A76], under some conditions on $g$, this relative asymptotic behavior was studied for the case of sequences of polynomials orthogonal with respect to varying measures. Moreover, in [A48] new results were obtained for the case of two fixed measures $d\mu$ and $d\rho$ supported on an arc of the unit circle.

In [A24] and [A28], the analysis was done assuming conditions of admissibility. In [A24, Theorem 2.1, pg. 21], the authors took advantage of the methods used in [55], generalizing them to the case of varying measures. The relative asymptotics of the polynomials $\varphi_{n,m}$ was determined on compact subsets of the region $\{|z| > 1\}$. In this way, the results obtained generalize those in [55] corresponding to fixed measures.

In [A28, Theorem 1.1, pg. 142], if $g$ satisfies a Lipschitz condition on a certain subset of $\Gamma$ the relative asymptotics is derived on this subset of $\Gamma$. The corresponding result for fixed measures was obtained in [68].

Once again, the results obtained in [A32] for varying measures on the unit circle were applied to the real case. Under weaker assumptions than those imposed in [A24] and [A28], in [A32, Theorem 1, pg. 510] the relative asymptotics on $\{|z| > 1\}$ was analyzed and in [A32, Theorem 2, pg. 517] the relative asymptotics on arcs $\gamma \subset \Gamma$ was studied.

A new kind of varying measures $\{d\rho_n\}$ was introduced in [A57]. These measures, used in [A57] and in subsequent papers, were defined as $d\rho_n = d\widetilde{\rho}_n/|W_n|^2$, where $\{d\widetilde{\rho}_n\}_{n\in\mathbf{N}}$ is a sequence of finite Borel measures on $\Gamma$. This concept generalizes the cases studied in previous papers, where $d\rho_n = d\rho$ was fixed. Also, a new concept of admissibility, called *weak admissibility*, was introduced in [A57, Definition 1, pg. 556], generalizing the concept of admissibility used previously.

Assuming this new concept, in [A57, Theorem 1, pg. 556] the weak convergence of varying measures was obtained. That result extends an analogous one given in [A31, Theorem 1, pg. 200]. Moreover, in [A57, Definition 2, pg. 556] a stronger concept of admissibility was introduced. With this additional hypothesis, in [A57, Theorem 3, pg. 557] the ratio asymptotic behavior was given extending [A31, Theorem 3, pg. 201]. As in previous papers, the keys for proving Theorem 3 were the weak convergence and the convergence obtained in [A57, Theorem 2, pg. 557] under the restriction given by the strong admissibility.

In [A52] the hypothesis of admissibility introduced in [A57, Definition 1, pg. 556] was assumed. Under certain additional condition, for a measure $\rho$ in the Szegö class the strong asymptotic behavior of orthogonal polynomials with respect to varying measures was given in [A52, Theorem 1, pg. 93]. As in [A57], the weak convergence was the main tool for proving this Szegö type result.

The concepts of admissibility introduced in [A57] and the result obtained in that paper were used in [A64] for studying relative asymptotics. In this way, [A64, Theorem 3.1, pg. 291] extends the results of relative asymptotics obtained in [A24], [A28], and [A32].

The paper [A48] deals with asymptotic formulas for sequences of polynomials orthogonal with respect to measures supported on an arc $\gamma \subset \Gamma$. The main contribution is [A48, Theorem 1, pg. 218], where Rakhmanov's theorem on ratio asymptotics is extended (see [66] and [67]). The condition $\rho' > 0$ a.e. on $\Gamma$ assumed in Rakhmanov's result was replaced by $\rho' > 0$ a.e. on $\gamma \subset \Gamma$. The convergence was proved on compact subsets of $\mathbb{C} \setminus \gamma$. In this sense, it is a very relevant fact that, when $\gamma \neq \Gamma$, it was proved that the zeros of the orthogonal polynomials approach $\gamma$. However, if $\gamma = \Gamma$ the set of zeros may be dense in $\{z : |z| < 1\}$. Then, in this case it is only possible to determine the convergence of the ratio on the unbounded component of $\mathbb{C} \setminus \gamma$.

In the proof of [A48, Theorem 1], varying measures were used again. In this case, the idea was to translate the problem on the unit circle in terms of varying measures on the real line. In [A48, Lemma 1, pg. 221] and [A48, Lemma 2, pg. 222] some interesting formulas were proved, establishing relations between sequences of polynomials orthogonal with respect to varying measures on the real line and sequences of polynomials orthogonal on $\Gamma$. Moreover, in [A48, Lemma 3, pg. 224], assuming $\rho' > 0$ a.e. on $\gamma$, the relative asymptotic behavior between the two families of polynomials was derived.

The second main result in [A48] is Theorem 2, in page 220. There, the relative asymptotic behavior of two sequences of orthogonal polynomials on an arc $\gamma \subset \Gamma$ was studied. One of these sequences is given by the polynomials orthogonal with respect to a measure $d\rho$ such that $\rho' > 0$ a.e. on $\gamma$. The second measure of orthogonality is $hd\rho$, where $h$ is a nonnegative integrable function with respect to $\rho$ such that there exists a polynomial $Q$ for which $Qh\,,\, Qh^{-1} \in L_\infty(\rho)$. The convergence was obtained on compact subsets of $\bar{\mathbb{C}} \setminus \gamma$. In this way, the corresponding result by A. Maté, P. Nevai, and V. Totik given in [56] was extended. (In [56], under the assumption $\rho' > 0$ a.e. on $\Gamma$, the convergence was obtained on the unbounded component of $\bar{\mathbb{C}} \setminus \Gamma$.)

On the other hand, [A48, Theorem 2] complements an analogous result obtained in [A32, Theorem 1] in the case of varying measures.

In [A48, Remark 3, pg. 231] the existence of a class on the unit circle analogous to the Blumenthal-Nevai class was conjectured. This problem was solved in [A53], where the new class $M_\Gamma(a;b)$ of measures on the unit circle was introduced. This class was defined in terms of the limit behavior of $|\Phi_n(0)|$ and $\Phi_{n+1}(0)/\Phi_n(0)$, where $\{\Phi_n(0)\}_{n\in\mathbf{N}}$ is the sequence of reflection coefficients for the monic polynomials $\{\Phi_n\}_{n\in\mathbf{N}}$ orthogonal with respect to the measure $\mu$. After this work, the class $M_\Gamma(a;b)$ was called the *López class*.

As in the real case, the measures $\mu$ of $M_\Gamma(a;b)$ were characterized in terms of the asymptotic behavior of the ratio $\Phi_{n+1}(z)/\Phi_n(z)$. This fact was established in [A53, Theorem 1, pg. 3].

More generally, $M_\Gamma(a_1,\ldots,a_k;b_1,\ldots,b_k)$ was defined, for any $k \in \mathbb{N}$, as the class of measures for which

$$\lim_{m\to\infty} |\Phi_{mk+i}(0)| = a_i\,, \quad \lim_{m\to\infty} \frac{\Phi_{mk+i}(0)}{\Phi_{mk+i-1}(0)} = b_i\,, \quad i = 1,\ldots,k$$

(see [A53, Definition 1, pg. 2]). In [A53, Theorem 2, pg. 14] and [A53, Theorem 3, pg. 15] this class was characterized as the set of measures whose polynomials have periodic ratio asymptotics. For each measure $\mu \in M_\Gamma(a_1,\dots,a_k;b_1,\dots,b_k)$ the limit of the ratio was determined. Moreover, it was proved that the convergence is uniform on compact sets of $\mathbb{C}\setminus(E\cup \text{supp }\mu)$. The derived set $(\text{supp }\mu)'$ of supp $\mu$ consists of the union of a finite number of arcs, supp $\mu\setminus(\text{supp }\mu)'$ is at most countable without accumulation points in $\Gamma\setminus\text{supp}\mu$, and $E$ contains at most a finite number of points which depend on the geometry of the problem. As a consequence of Theorems 2 and 3 of [A53], the asymptotic behavior of the zeros of the orthogonal polynomials was described for each measure $\mu \in M_\Gamma(a_1,\dots,a_k;b_1,\dots,b_k)$ .

Theorems 2 and 3 of [A53] were obtained under the assumption $a_i \in (0,1)\,,\, i = 1,\dots,k$. Moreover, in [A53, Remark 4, pg. 28] the limit case $a_i = 1$ for some $i \in \{1,\dots,k\}$ was analyzed.

In [A63] the class $M_\Gamma(a_1,\dots,a_k;b_1,\dots,b_k)$ was extended to the case $a_i \in [0,1]$. If $a_i = 0$ for some $i \in \{1,\dots,k\}$, then it is easy to see $a_1 = \cdots = a_k = 0$; that is, $\lim_n \Phi_n(0) = 0$. In [A63, Theorem 1, pg. 173] the asymptotic behavior of the ratio $\Phi_{n+k}(z)/\Phi_n(z)$ was determined.

Let $\mu$ be a measure on $\Gamma$ satisfying Szegö's condition. In [A63, Theorem 2, pg. 174] a condition, which guarantees $\mu \in M_\Gamma(a;b)$, was given. That condition was formulated in terms of the inner Szegö function, without using the reflection coefficients.

The paper [A69] is an example of an application of polynomials orthogonal on the unit circle. Let $f(z) = \sum_{n=0}^\infty f_n z^n$ be an analytic function in some neighborhood of $z = 0$. Let $R_m(f) > 0$ be the radius of the largest disk $D_m := \{z : |z| < R_m(f)\}$ where $f$ has at most $m$ poles. In [42], the following result for determining $R_m(f)$ was proved.

**Theorem 4.1 (Hadamard)** *For each $m \geq 0$ we have $R_m(f) = l_m/l_{m+1}$, where*

$$l_m = \limsup_n |H_{n,m}|^{1/n}\,, \quad H_{n,m} = \begin{vmatrix} f_{n-m+1} & f_{n-m+2} & \cdots & f_n \\ f_{n-m+2} & f_{n-m+3} & \cdots & f_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_n & f_{n+1} & \cdots & f_{n-m+1} \end{vmatrix}\,, \quad n \geq m-1\,.$$

In [A69, Theorem 1, pg. 264] an analogue of Hadamard's Theorem for analytic functions on $\{z : |z| \leq 1\}$ was proved. Measures for which $\limsup_n |\varphi_n(0)|^{1/n} < 1$ were used. The proof was based on results by S. P. Suetin on the convergence of rows of Fourier-Padé approximantion of analytic functions on the closed unit circle (see [74] and [75]).

# 5 Orthogonal polynomials and linear operators theory.

It is well known the relation between the linear operators theory and the study of sequences of polynomials orthogonal with respect to measures supported on the real line. The connection between both topics is given by the three-term recurrence relation

$$\left.\begin{aligned} P_{n+1}(z) &= (z-b_n)P_n(z) - a_n^2 P_{n-1}(z) \quad , \quad n = 0,1,\dots \\ & P_0(z) \equiv 1 \quad , \quad P_{-1}(z) \equiv 0\,, \end{aligned}\right\} \tag{15}$$

whose coefficients $a_n$ , $b_n$ , $n \in \mathbb{N}$ , define a tridiagonal (Jacobi) matrix

$$G = \begin{pmatrix} b_0 & a_1 & \\ a_1 & b_1 & \ddots \\ & \ddots & \ddots \end{pmatrix} . \tag{16}$$

On the other hand, when $a_n, b_n \in \mathbb{R}$ , $n \in \mathbb{N}$ , the properties of the continued fraction

$$\cfrac{a_0^2}{z - b_0 - \cfrac{a_1^2}{z - b_1 - \cfrac{a_2^2}{z - b_2 - \ddots}}}, \qquad a_n \neq 0, \quad n \in \mathbb{N}, \tag{17}$$

can be analyzed by studying the Jacobi matrix (16). In this way, the theory of linear operators is a useful tool in approximation theory.

In the papers [A40], [A45], [A49], and [A54] the connection between sequences of polynomials given by a three-term recurrence relation and operators was used, extending some classical results to the case when $G$ is a symmetric complex matrix (i.e., $G = G^T$), whose entries are, in general, $a_n, b_n \in \mathbb{C}$. In these papers, the decomposition

$$G = H + C \tag{18}$$

was used, being $H, C$ the matrix representations of a self-adjoint operator and a bounded operator respectively. In particular, in [A40] and [A45], $H = \mathfrak{Re}\, G$ , $C = i\mathfrak{Im}\, G$ was assumed. In both papers, the main idea was to study the spectrum of $G$ under certain restrictions. The conditions

$$\lim_n \mathfrak{Im}\, a_n = \lim_n \mathfrak{Im}\, b_n = 0\,, \quad \sup_n\{|a_n|, |b_n|\} < \infty \tag{19}$$

were assumed.

In [A40, Theorem 2, pg. 312], under the above conditions, the asymptotic behavior of zeros of $\{P_n\}_{n\in\mathbf{N}}$ was stated. As a consequence, in [A40, Theorem 1, pg. 311] the convergence of the continued fraction (17) was deduced.

When $a_n$ , $b_n \in \mathbb{R}$ , the sequence $\{P_n\}_{n\in\mathbf{N}}$ of monic polynomials generated by (15) belongs to the Blumenthal-Nevai class if and only if

$$\exists \lim_n a_n = a\,, \quad \exists \lim_n b_n = b\,. \tag{20}$$

This class is constituted by the sequences of polynomials for which there exists the ratio asymptotics (see [59]). In [A40, Definition 1, pg. 312] the Blumenthal-Nevai class was extended for polynomials generated by recurrence relations (15) with complex coefficients. In [A40, Theorem 3, pg. 312] this class was characterized in terms of the ratio asymptotic behavior, extending the corresponding result given in [59] for the real case.

In [A45] the condition (20) was replaced by

$$\exists \lim_m a_{mk+i} = a^{(i)} \quad \exists \lim_m b_{mk+i} = b^{(i)}\,, \quad i = 0, 1, \ldots, k-1\,, \tag{21}$$

for some $k \in \mathbb{N}$. Under this generalization, the main results given in [A40] were improved. In particular, in [A45, Theorem 1, pg. 633] the convergence of the continued fraction (19) was obtained in a more extended region than the given in [A40, Theorem 1]. Moreover, in [A45, Theorem 2, pg. 633] the asymptotic behavior of the ratio of polynomials $\{P_n\}_{n\in\mathbf{N}}$ was studied, extending Theorem 3 of [A40]. On the other hand, [A45, Theorem 2] extends, also, the corresponding results given in [35] and [79] in the case of asymptotically periodic coefficients $a_n, b_n \in \mathbb{R}$. As in [A40], the location of zeros of $\{P_n\}_{n\in\mathbf{N}}$, given in [A45, Theorem 3, pg. 639], was the key to obtain the rest of results.

In [A49] and [A54] some results obtained in [A40] and [A45] were extended to the case that $G$ represents a non necessarily bounded operator. In [A49] it was assumed that $H$ represents in (18) a self-adjoint operator. In [A49, Definition 1, pg. 181] the concepts of *determinate case* and *indeterminate case* for $G$ were introduced. This definition was given in terms of the sequence $\{P_n\}$ of polynomials generated by (15). This concept extends the corresponding definition in the case that the entries of $G$ are real numbers and the polynomials given in (15) are orthogonal with respect to some real measure. The main result in [A49] is Theorem 2, in page 193, where the restriction

$$\lim_n \mathfrak{Im} a_n = \lim_n \mathfrak{Im} b_n = 0$$

was assumed. Here if $G$ is determinate, then the convergence of the continued fraction (17) was obtained on compact sets of the region $\mathbb{C} \setminus (\mathbb{R} \cup \sigma_p(G))$ ($\sigma_p(G)$ being the point spectrum of $G$). Thus, [A49, Theorem 2] extends the well-known Stieltjes' Theorem related to the case $a_n, b_n \in \mathbb{R}$ (see [73]). As in other results given in [A40] and [A45], the main tool in the proof of [A49, Theorem 2] was the analysis of the spectrum of $G$. It was proved that the continued fraction (17) converges to a function $f(z)$ meromorphic in $\mathbb{C} \setminus \mathbb{R}$, each $z \in \sigma_p(G) \setminus \mathbb{R}$ being a pole of $f(z)$. In this paper, several results of independent interest, related to the spectrum of $G$, were obtained. Moreover, in [A49, Lemma 1, pg. 182] a sufficient Carleman–type condition for the determinate case was given.

In [A54], as in [A49], bounded perturbations of a self-adjoint operator, given by (18), were studied. In this paper, some results were obtained in the case that $G$ is a band matrix, the operator $H$ in (18) not being necessarily bounded.

It is possible to identify the finite matrix $G_n$, formed by the first $n$ rows and the first $n$ columns of $G$, with the infinite matrix

$$\begin{pmatrix} G_n & 0 \\ 0 & 0 \end{pmatrix}.$$

With this premise, in [A54, Theorem 1, pg. 503] the strong convergence of the sequence of resolvent operators $\{(G_n - zI)^{-1}\}$ was analyzed. For this purpose, in [A54, Lemma 2, pg. 504] and [A54, Lemma 3, pg. 505], some bounds for the spectra of operators $G_n$, $n \in \mathbb{N}$, and $G$ were obtained. Moreover, in [A54, Lemma 3] a uniform bound of the sequence $\{(G_n - zI)^{-1}\}$ was obtained.

Set $e_0 = (1, 0, \ldots) \in \ell^2$, the relation

$$f_n(z) = -\langle (G_n - zI)^{-1} e_0, e_0 \rangle \,, \quad n \in \mathbb{N} \,,$$

between the convergents $f_n(z)$ of (17) and the resolvent operators $(G_n - zI)^{-1}$ (see, for instance, [14]) is well known . From this and [A54, Theorem 1], in [A54, Theorem 2, pg. 511] the convergence of the continued fraction (17) was deduced.

In [A40], [A45], [A49], and [A54], the matrices $G^{(k)}$, obtained from $G$ by deleting its first $k$ rows and $k$ columns, were used. In [A54, Lemma 6, pg. 508], under the assumption that $H$ is self-adjoint, it was proved that $H^{(k)}$ is also self-adjoint. Moreover, if $C$ is a compact operator, then in (18) the decomposition

$$G^{(k)} = H^{(k)} + C^{(k)}$$

is possible. In these conditions, the spectrum of $G^{(k)}$ is "near" to the spectrum of $H^{(k)}$. This fact was the main idea in the proof of [A54, Theorem 3, pg. 512], where the strong convergence of the sequence $\{(G_n - zI)^{-1}\}$ to $(G - zI)^{-1}$ was obtained. In this result, the region of convergence is $\mathbb{C} \setminus (\sigma_p(G) \cup S)$, where $S \subset \mathbb{R}$.

As a consequence of Theorem 3, in [A54, Corollary 6, pg. 516] the above mentioned Stieltjes' Theorem was extended. Moreover, in [A54, Theorem 4, pg. 517] the poles of the limit function $f(z) = -\langle (G - zI)^{-1} e_0, e_0 \rangle$ were located in $\sigma_p(G) \setminus S$.

Given a system of Borel measures $\{\mu_i\}$ , $i = 1, \ldots, m$ , Supp $\mu_i \subset \mathbb{R}$ , and given a multi-index $\mathbf{n} = (n_1, \ldots, n_m)$ , $n_i \in \mathbb{Z}_+$ , it is possible to find a (monic) multiple orthogonal polynomial $P_{\mathbf{n}}$ (not necessarily unique), deg $P_{\mathbf{n}} \leq n_1 + \cdots n_m$ , such that

$$\int x^\nu P_{\mathbf{n}}(x) d\mu_i(x) = 0 \,, \quad \nu = 0, \ldots, n_i - 1 \,, \quad i = 1, \ldots, m \,.$$

Assume that the set of multi-indices $I = \{\mathbf{n} = (n_1, \ldots, n_m) : \, n_i \geq 0 \,, n_1 \geq n_2 \geq \cdots \geq n_m \geq n_1 - 1\}$ is used. If $\mathbf{n}$ is normal for all $\mathbf{n} \in I$ (i.e., $\deg P_{\mathbf{n}} = n_1 + \cdots + n_m$), then the sequence of polynomials $\{P_{\mathbf{n}}\}_{\mathbf{n} \in \mathbf{N}^m}$ satisfies a $(m+2)$- terms recurrence relation,

$$P_{\mathbf{n+1}}(x) = (x - b_{nn}) P_{\mathbf{n}}(x) + b_{n,n-1} P_{\mathbf{n-1}}(x) + \cdots + b_{n,n-m} P_{\mathbf{n-m}}(x)$$

with initial conditions $P_{-\mathbf{m}}(x) = \cdots = P_{-\mathbf{1}}(x) = 0 \,, P_{\mathbf{0}}(x) = 1$. In a more general situation, in [A77, Section 4.1] the concept of *complete totally ordered sequence of multi-indices* was introduced. In [A77, Proposition 4.1, pg. 361] was proved that, for this kind of multi-indices with an additional condition, there exists a $p \in \mathbb{N}$ such that the corresponding sequences of multiple orthogonal polynomials $\{P_{\mathbf{n}}\}$ satisfy $(p+2)$-term recurrence relation. This recurrence relation is associated with the $(p+2)$-banded matrix

$$\widetilde{G} = \begin{pmatrix} \alpha_{00} & 1 & 0 & & \\ \alpha_{10} & \alpha_{11} & 1 & \ddots & \\ & \alpha_{21} & \ddots & \ddots & \\ \vdots & & \ddots & & \\ \alpha_{m0} & \vdots & & & \\ 0 & \alpha_{m1} & & & \\ & 0 & \ddots & & \\ & & \ddots & & \end{pmatrix}.$$

As in the classical case of three-term recurrence relations and tridiagonal matrices, the properties of the spectrum of $\widetilde{G}$ are related with the properties of the sequence $\{P_{\mathbf{n}}\}_{\mathbf{n}\in\mathbf{N}^m}$. From this fact, it is possible to deduce some consequences for $\widetilde{G}$ and its spectrum using the obtained results for the multiple orthogonal polynomials. In particular, in [A77, Theorem 1.1, pg. 349] it was proved that, under certain conditions, $\widetilde{G}$ represents a bounded operator if and only if the zeros of $\{P_{\mathbf{n}}\}$ lie on a real bounded interval. With the same premises that in Theorem 1.1, in [A77, Theorem 1.2, pg. 350] the asymptotic behavior of the ratio was studied in terms of the periodic limit of the diagonals entries $\{\alpha_{k+i,i} : i = 0, 1, \ldots\}$ of $\widetilde{G}$. In this way, [A45, Theorem 2, pg. 633] was extended for this kind of multiple orthogonal polynomials.

Let $L^2_\mu$ be the space of square integrable functions with respect to a measure $\mu$, supported on the unit circle. An important tool in the study of the sequence of polynomials orthogonal with respect to $\mu$ is the unitary multiplication operator $U : L^2_\mu \longrightarrow L^2_\mu$,

$$(Uf)(z) = zf(z), \quad f \in L^2_\mu.$$

Let $\{\Phi_n\}_{n\in\mathbf{N}}$ be the corresponding sequence of monic orthogonal polynomials. It is well known that, if Szegö's condition is not verified (i.e., $\log \mu' \notin L^1_\mu$), then the matrix

$$U_\mu = \begin{pmatrix} -\Phi_1(0)\overline{\Phi_0(0)} & -\frac{\kappa_0}{\kappa_1}\Phi_2(0)\overline{\Phi_1(0)} & -\frac{\kappa_0}{\kappa_2}\Phi_3(0)\overline{\Phi_0(0)} & \cdots \\ \frac{\kappa_0}{\kappa_1} & -\Phi_2(0)\overline{\Phi_1(0)} & -\frac{\kappa_1}{\kappa_2}\Phi_3(0)\overline{\Phi_1(0)} & \cdots \\ 0 & \frac{\kappa_1}{\kappa_2} & -\Phi_3(0)\overline{\Phi_2(0)} & \cdots \\ 0 & 0 & \frac{\kappa_2}{\kappa_3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

represents the operator $U$ when the basis of orthonormal polynomials is used in $L^2_\mu$ (here, $\kappa_m$ is the leading coefficient for the $m$-th orthonormal polynomial). Moreover, for each polynomial $\Phi_n(z)$ the set of zeros coincides with the spectrum of the finite matrix $U_{\mu,n}$, given by the first $n$ columns and the first $n$ rows of $U_\mu$ (see [69]). In other words, the sequence $\{\Phi_n\}_{n\in\mathbf{N}}$ satisfies

$$\Phi_{n+1}(z) = (z + \Phi_{n+1}(0)\overline{\Phi_n(0)})\Phi_n(z) + \sum_{i=0}^{n-1} \left(\frac{\kappa_i}{\kappa_n}\right)^2 \Phi_{n+1}(0)\overline{\Phi_i(0)}\Phi_i(z).$$

In [A53], another well-known recurrence relation was used,

$$\Phi_n(0)\Phi_{n+1}(z) - (z\Phi_n(0) + \Phi_{n+1}(0))\,\Phi_n(z) + z\frac{\kappa_{n-1}^2}{\kappa_n^2}\Phi_{n+1}(0)\Phi_{n-1}(z) = 0\,, \quad n \geq 2$$

(see [A53, Lemma 2, pg. 5]). Using this three-term recurrence relation, in [A53, Lemma 3, pg. 7] the expression of $\Phi_n(z)$ in terms of the leading principal submatrices

$$D = \begin{pmatrix} z + \frac{\Phi_1(0)}{\Phi_0(0)} & z\frac{\Phi_2(0)}{\Phi_1(0)}\left(\frac{\kappa_0}{\kappa_1}\right)^2 & 0 & \cdots \\ 1 & z + \frac{\Phi_2(0)}{\Phi_1(0)} & z\frac{\Phi_3(0)}{\Phi_2(0)}\left(\frac{\kappa_1}{\kappa_2}\right)^2 & \cdots \\ 0 & 1 & z + \frac{\Phi_3(0)}{\Phi_2(0)} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{22}$$

was established. In the section dedicated to the orthogonal polynomials on the unit circle, the main results obtained in [A53] were discussed. Here, we only want to underline that the study of the spectrum of $U_\mu$, and also of the spectrum of the finite sections of (22), was the main tools in the proof of those results.

## 6 Hermite-Padé approximants.

Simultaneous rational approximation, specifically the Hermite-Padé approximation for systems of Cauchy transforms, has been one of the favorite subjects within Guillermo's wide research. Setting aside their own interest, the Hermite-Padé approximation and its akin topic, the multi-orthogonal polynomials, have been useful to obtain a broad range of results in number theory (cf. [60, 71, 8, 27]). They also have applications in numerical analysis, specially to quadratures (cf. [9, 12, 20, A81]), and in other directions like random matrices [11, 6] and Brownian motion [21]. For information about the Hermite-Padé approximation, see the surveys [63, 7, 4] and the book [62].

The proof of the transcendence of the number $e$ by Hermite in 1873 marks the beginning of the history of the Hermite-Padé approximation. In [43], he introduced a kind of simultaneous rational approximants for systems of exponential functions, where each approximant is a vector of rational functions with the same denominator. Such rational vectors are nowadays called Hermite-Padé approximants. Two different lines of research can be distinguished in the theory: the formal or algebraic direction and the analytic or asymptotic one. The former is concerned with the algebraic properties of the approximants such as uniqueness and aims at the study of particular systems of functions like exponentials, logarithms, etc... The latter has to do with the description in several senses of the behavior of the approximants for general classes of systems as the degree of the common denominator tends to infinity.

In the sixties, several authors such as Mahler [53], Coates [19], and Jager [45] focused on the algebraic aspects whereas the interest for the analytic side grew in the eighties due to works by Nikishin [61], Kalyagin [47], and Gonchar and Rakhmanov [38], among others. Over the last fifteen years, Guillermo has constantly contributed to both sides of the topic. He has obtained uniqueness conditions for the approximants for a wide class of multi-indices, results on denominator's zeros location, conditions for convergence in several senses, such as weak, ratio, strong convergence, convergence in capacity, etc...

In order to summarize Guillermo's contributions, let us give some definitions. Rather than seeking the most general situation, we will focus our attention on systems formed by Cauchy transforms. Fix a multi-index $\mathsf{n} = (n_1, \dots, n_m) \in \mathbf{Z}_+^m$, $\mathbf{Z}_+ = \{0, 1, \dots\}$, and a system of functions $\widehat{S} = (\widehat{s}_1, \dots, \widehat{s}_m)$ given by

$$\widehat{s}_i(z) = \int \frac{ds_i(x)}{z - x}, \quad i = 1, \dots, m,$$

where $S = (s_1, \dots, s_m)$ is a system of finite Borel measures with constant sign. The support $\Sigma_i$ of each measure $\sigma_i$, $1 = 1, \dots, m$, is a set in the real line, not necessarily bounded, consisting of infinitely many points. The smallest interval containing $\Sigma_i$ will be denoted by $\Delta_i$, $i = 1, \dots, m$. A vector of rational functions

$$R_{\mathsf{n}} = (R_{\mathsf{n},1}, \dots, R_{\mathsf{n},m}) = (P_{\mathsf{n},1}/Q_{\mathsf{n}}, \dots, P_{\mathsf{n},m}/Q_{\mathsf{n}}),$$

is called an Hermite-Padé approximant corresponding to $\widehat{S}$ and $\mathbf{n}$, if

$$\deg Q_{\mathsf{n}} \le |\mathbf{n}| = n_1 + \cdots + n_m, \quad Q_{\mathsf{n}} \not\equiv 0, \quad \deg P_{\mathsf{n},i} \le |\mathsf{n}| - 1, \quad i = 1, \dots, m,$$

and the following interpolating relations at infinity are satisfied

$$Q_{\mathsf{n}}(z)\widehat{s}_i(z) - P_{\mathsf{n},i}(z) = \mathcal{O}\left(\frac{1}{z^{n_i+1}}\right), \quad z \to \infty, \quad i = 1, \dots, m. \tag{23}$$

The polynomial $Q_{\mathsf{n}}$ is taken to be monic. When the functions $\{\widehat{s}_i\}$ are interpolated at a table of points, $R_{\mathsf{n}}$ is called a multi-point Hermite-Padé approximant. If some zeros of $Q_{\mathsf{n}}$ are fixed beforehand, $R_{\mathsf{n}}$ is then a generalized Hermite-Padé approximant. Overall, the name Hermite-Padé approximation comprises all these concepts.

For each multi-index $\mathbf{n}$, the common denominator $Q_{\mathsf{n}}$ turns out to be a multi-orthogonal polynomial with respect to the system of measures $S$ since it satisfies the orthogonality relations

$$\int x^\nu Q_{\mathsf{n}}(x)\, ds_i(x) = 0, \quad \nu = 0, 1, \dots, n_i - 1, \quad i = 1, \dots, m. \tag{24}$$

It is not so hard to see that finding a polynomial $Q_{\mathsf{n}} \not\equiv 0$ from (24) reduces to solving a system of $|\mathbf{n}|$ homogeneous linear equations for $|\mathbf{n}| + 1$ unknown coefficients of $Q_{\mathsf{n}}$. The quantities $P_{\mathsf{n},i}$, $i = 1, \dots, m$, are the polynomial parts of the power expansions of the products $Q_{\mathsf{n}}\,\widehat{s}_i$ at infinity. Hence, there always exists a vector $R_{\mathsf{n}}$ which is univocally determined by $Q_{\mathsf{n}}$. Given $\widehat{S}$ and $\mathbf{n}$, the uniqueness of $R_{\mathsf{n}}$ is equivalent to the fact that the system of homogeneous linear equations deduced from (24), or from (23), has a solution space of dimension one. In general such a situation does not happen. For instance, if there exist $i, k \in \{1, \dots, m\}$, $i \ne k$, such that $s_i = s_k$ and $n_i, n_k > 0$. This is in striking contrast to the classical Padé approximation which corresponds to the case $m = 1$. In that case, $R_{\mathbf{n}} = P_{\mathbf{n},1}/Q_{\mathbf{n}}$ is exactly the $n_1$-diagonal Padé approximant of the function $\widehat{s}_1$, which is well known to be unique. Furthermore, conditions (24) yield $Q_{\mathbf{n}}$ to have exactly $|\mathbf{n}| = n_1$ simple zeros lying into $\Delta_1$. These nice properties are useful in obtaining results such as Markov's Theorem and Stieltjes' Theorem (see [54] and [73]) which assure uniform convergence of $R_{\mathbf{n}}$ to $\widehat{s}_1$ on any compact subset of $\mathbf{C} \setminus \Delta_1$.

A multi-index $\mathbf{n}$ is said to be normal if any non trivial solution $Q_{\mathbf{n}}$ of (24) satisfies $\deg Q_{\mathbf{n}} = |\mathbf{n}|$. If $Q_{\mathbf{n}}$ has exactly $|\mathbf{n}|$ simple zeros and they all lie in the interior of the smallest interval containing $\cup_{i=1}^m \Delta_i$ the multi-index is called strongly normal. Obviously, strong normality implies normality, and it is not difficult to see that normality implies that $Q_{\mathbf{n}}$ is uniquely determined (see [A68]). When all the multi-indices are either normal or strongly normal the system $S$ is said to be either perfect or strongly perfect, respectively.

As can be easily guessed, normality of multi-indices plays a fundamental role in the study of the convergence of the Hermite-Padé approximants. The fact that a multi-index may not be normal shows how much harder to obtain results in simultaneous approximation is as compared to classical Padé approximation. For that reason, research on asymptotics has been focused on particular systems of Cauchy transforms for which normality can be proved for a wide choice of multi-indices: the so-called Angelesco and Nikishin systems.

A system of functions $\widehat{S}$ is called an Angelesco system if $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$ (see [2]). The Angelesco systems are strongly perfect. The first result on convergence of simultaneous Padé approximants for these systems is due to Kalyagin [47] who considered $m = 2$, $\mathbf{n} = (n, n)$, and particular choices of $\{\Delta_i\}$ and $\{s_i\}$ managing to prove geometric rate of convergence of $R_{\mathbf{n},i}$ to $\widehat{s}_i$, $i = 1..2$, outside the set $\Delta_1 \cup \Delta_2$.

A major result on Angelesco systems was obtained by Gonchar and Rakhmanov [38] under the assumption $ds_i(x)/dx > 0$ a. e. on $\Delta_i$, $i = 1, \ldots, m$. As regards the multi-indices, they require that the orthogonality relations are proportionally distributed among the components of $\mathbf{n}$, that is, $\lim_{|\mathbf{n}| \to \infty} n_i/|\mathbf{n}| = c_i > 0$, $i = 1, \ldots, m$. They use a vector potential equilibrium problem to describe the $n$th-root asymptotics of $Q_{\mathbf{n}}$ and the limit distributions of their zeros on the intervals $\Delta_i$. The distinctive feature is that the supports of the vector equilibrium measure are intervals $\Delta_i^* \subset \Delta_i$ not coinciding in general with the system of intervals $\{\Delta_i\}$. As a consequence, domains of divergence of $R_{\mathbf{n}}$ may appear in $\mathbf{C} \setminus \cup_{i=1}^m \Delta_i$ depending on the geometry of the intervals $\Delta_i$ and the numbers $c_i$, $i = 1, \ldots, m$, but not on the system of measures $S$.

The strong asymptotic behavior of $Q_{\mathbf{n}}$ was studied by Aptekarev in [3] for $m = 2$ and $\mathbf{n} = (n, n)$. The natural hypothesis is that the involved measures belong to the Szegő class.

The Nikishin systems, introduced by Nikishin in [61], are, in a sense, opposite to the Angelesco systems since all the measures of $S$ are now supported on the same interval. In fact, Gonchar, Rakhmanov, and Sorokin by using tree graphs introduced in [40] systems of Cauchy transforms, called generalized Nikishin systems, which contain Angelesco and Nikishin systems as particular and extreme cases. Let us adopt their notation. Take two disjoint intervals $\Delta_1$ and $\Delta_2$ in $\mathbf{R}$ and two measures $\sigma_1$ and $\sigma_2$ with supports in $\Delta_1$ and $\Delta_2$ respectively. Define the following kind of product of measures

$$d\langle\sigma_1,\sigma_2\rangle(x) = \int \frac{d\sigma_2(t)}{x-t}\, d\sigma_1(x) = \widehat{\sigma}_2(x) d\sigma_1(x),$$

so that $\langle\sigma_1,\sigma_2\rangle$ is a measure with support in $\Delta_1$. Given now a system of intervals $\Delta_1,\ldots,\Delta_m$ satisfying $\Delta_{i-1}\cap\Delta_i\emptyset$, $i=2,\ldots,m$, and measures $\sigma_1,\ldots,\sigma_m$, where the support of $\sigma_i$ is contained in $\Delta_i$, $i=1,\ldots,m$, let us define inductively

$$\langle\sigma_1,\sigma_2,\ldots,\sigma_j\rangle = \langle\sigma_1,\langle\sigma_2,\ldots,\sigma_j\rangle\rangle, \quad j=1,\ldots,m.$$

Set $s_j = \langle\sigma_1,\sigma_2,\ldots,\sigma_j\rangle$, $j=1,\ldots,m$. The system of Cauchy transforms

$$\widehat{S} = (\widehat{s}_1,\ldots,\widehat{s}_m) = N(\sigma_1,\ldots,\sigma_m)$$

corresponding to the system of measures $S=(s_1,\ldots,s_m)$ is the Nikishin system generated by the measures $\{\sigma_1,\ldots,\sigma_m\}$.

The question of whether the Nikishin systems are strongly perfect is still an open problem. Nevertheless, the class of multi-indices for which strong normality holds has been widening over the years due to the efforts of several people. The first who found strongly normal indices for such systems was Nikishin himself. In [61], he proved that all multi-indices of the form

$$\mathbf{n} = (n_1,\ldots,n_m) \text{ where } n_i = \begin{cases} n+1, & i\le q; \\ n, & i>q, \end{cases} \qquad 0\le q\le m, \tag{25}$$

are strongly normal. Also, without the need of imposing any condition on the measures $\sigma_i$, he obtained uniform convergence of $R_\mathbf{n}$ to $\widehat{S}$ on compact subsets of $\mathbf{C}\setminus\Delta_1$ provided $\Delta_1$ is bounded, $m=2$, and $\mathbf{n}=(n,n)$.

Analyzing Nikishin's result, Gonchar observed that a number $n$ of additional interpolation points automatically appear on $\Delta_2$ for each of the functions $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$, thus reducing the proof to the study of multipoint Padé approximation and polynomials orthogonal with respect to varying measures. Following Gonchar's approach, Nikishin's result was extended in [A35] to a system of arbitrary $m$ functions with strongly normal indices (25) without requiring $\Delta_1$ to be bounded, as long as $\Delta_2$ is bounded or Carleman's condition is fulfilled for the measure $s_1=\sigma_1$. A key step was to prove convergence in capacity of the approximants $R_\mathbf{n}$ to $\widehat{S}$ for which multi-indices were not required to be strongly normal but just that their components increased uniformly. The reference [A35] is fundamental because all the subsequent works dealing with asymptotics used the same strategy based on the fact that, for each $i=1,\ldots,m$, the approximant $R_{\mathbf{n},i}$ is close to being a multipoint Padé approximant of $\widehat{s}_i$. Actually, $\mathbf{n}+n_i+1$ interpolation conditions of $R_{\mathbf{n},i}$ are assigned at infinity by construction whereas a bonus of nearly $\mathbf{n}-n_i$ interpolation points is obtained on $\Delta_2$.

Some of the early consequences of [A35] (whose Russian version was published in 1992) are [A34], [A36], and [A41]. The rate of convergence of $R_{\mathbf{n},i}$ to $\widehat{s}_i$, $i=1,\ldots,m$, was estimated in [A34] and proved to be of geometric order. There, all the intervals $\Delta_i$ are assumed to be bounded and the multi-indices take the form (25). Nikishin systems of two functions perturbed with rational aggregates were considered in [A41]. In the spirit of Gonchar's work [36], the aim was to study the effect that the poles of the functions to be approximated produce on the poles of the simultaneous approximants. The multi-indices are of type $\mathbf{n} = (n, n)$ or $\mathbf{n} = (n, n+1)$ and the coefficients of the rational functions are real. As expected, the poles of the rational functions attract some of the poles of the approximants and uniform convergence of $R_{\mathbf{n},i}$ to $\widehat{s}_i$, $i = 1, \ldots, m$, is obtained under mild conditions. A similar problem was tackled in [A36], where for systems of $m$ meromorphic Nikishin type functions convergence in capacity is proved not only for the main diagonal of multi-indices but also for those sequences close to it.

The results in [A35] made evident the necessity of finding strongly normal multi-indices, because, according to Gonchar's Lemma [37], convergence in capacity of Hermite-Padé approximants becomes uniform convergence in the presence of strongly normal multi-indices. From the results in [A35], it follows immediately that the Nikishin systems made up by two functions are strongly perfect. A detailed proof of this was published by Driver and Stahl in [25] where they also extended a result that Nikishin stated without proof in [61]. Nikishin said that the decreasing indices are strongly normal, and they obtained such property being fulfilled by all indices satisfying $1 \leq j < k \leq m \Rightarrow n_k \leq n_j + 1$. That set has been denoted by $\mathbf{Z}_+^m(\circledast)$.

The logarithmic asymptotic behavior of $\{Q_{\mathbf{n}}\}_{\mathbf{n} \in \Lambda \subset \mathbf{Z}_+^m(\circledast)}$, where $\Lambda$ is such that the orthogonality relations are proportionally distributed among the components of $\mathbf{n}$ and $\sigma_i' > 0$ a. e. on $\Delta_i$, $i = 1, \ldots, m$, was obtained in [40]. The limit behavior is described in terms of a vector potential equilibrium problem as in the Angelesco case but the interaction between the components of the vector valued problem is different. Now, the supports of the vector equilibrium measure are the intervals $\Delta_i$, $i = 1, \ldots, m$. A key role in the proof is played by the functions of the second kind $\Psi_{\mathbf{n},i}$, $i = 1, \ldots, m$, defined inductively by

$$\Psi_{\mathbf{n},0}(z) = Q_{\mathbf{n}}(z), \quad \Psi_{\mathbf{n},i}(z) = \int \frac{\Psi_{\mathbf{n},i-1}(x)}{z - x} \, d\sigma_i(x), \quad i = 1, \ldots, m.$$

Denote by $Q_{\mathbf{n},i}$, $i = 1, \ldots, m$, the monic polynomial whose zeros are the zeros of $\Psi_{\mathbf{n},i-1}$ on $\Delta_i$. As a sophisticated generalization of the arguments used in [A35], each polynomial $Q_{\mathbf{n},i}$ turns out to be orthogonal with respect to a varying measure $\omega_{\mathbf{n},i}(x) \, d\sigma_i(x)$ with $\deg Q_{\mathbf{n},i} = n_i + \cdots + n_m$. Hence, $Q_{\mathbf{n},1} = Q_{\mathbf{n}}$. Additionally, the zero limit distributions of $Q_{\mathbf{n},i}$ are the components of the vector equilibrium measure. As a consequence, uniform convergence of $R_{\mathbf{n},i}$ to $\widehat{s}_i$, $i = 1, \ldots, m$, is obtained for compact subsets of $\mathbf{C} \setminus \Delta_1$. All the results of [40] are given in the more general setting of generalized Nikishin systems for which the above assertions must be adequately modified. For instance, in such a general situation domains of divergence for the simultaneous approximants do appear.

Concerning other kind of asymptotics of $\{Q_{\mathbf{n}}\}$ corresponding to Nikishin systems, weak star convergence was proved in [A57]. Let $q_{\mathbf{n},i}$ be the orthonormal polynomials corresponding to the varying measures $\omega_{\mathbf{n},i}(x) \, d\sigma_i(x)$. Suppose that $\sigma_i' > 0$ a. e. on $\Delta_i$ and $\mathbf{n} \in \Lambda \subset \mathbf{Z}_+^m(\circledast)$, where $\Lambda$ is a sequence of multi-indices such that $n_{i-1} - n_i \leq C$, $i = 2, \ldots, m$, with $C$ an absolute constant.

Then, the main result of [A57] is

$$q^2_{\mathbf{n},i}\,\omega_{\mathbf{n},i}\,\sigma_i \ \xrightarrow{\ *\ }\ \mu_{\Delta_i}, \quad n_1\to\infty, \quad i=1,\dots,m, \tag{26}$$

where $\mu_{\Delta_i}$ is the equilibrium measure of the interval $\Delta_i$. The proof of (26) relies on similar formulas for orthogonal polynomials with respect to a very general class of varying measures.

Strong convergence of $\{Q_{\mathbf{n}}\}$ for Nikishin systems was proved by Aptekarev in [5] for multi-indices of the form $\mathbf{n}=(n,\dots,n)$ and measures $\{\sigma_i\}$ in the Szegő class. Among other arguments, the formula (26) is used. Other related strong asymptotic formulas may be found in [A52] with the same choice of multi-indices as in [A57].

In [A73] ratio asymptotics of polynomials $\{Q_{\mathbf{n}}\}$ for Nikishin systems was obtained. To be more precise, it was shown that if $\sigma_i'>0$ a. e. on $\Delta_i$, $i=1,\dots,m$, then there exists a function $F$ analytic on $\mathbf{C}\setminus\Delta_1$ such that

$$\lim_{n\in\mathbf{I}}\frac{Q_{\mathbf{n+1}}(z)}{Q_{\mathbf{n}}(z)}=F(z), \qquad \mathbf{n}+\mathbf{1}=\mathbf{n}+(1,\dots,1), \tag{27}$$

uniformly on compact subsets of $\mathbf{C}\setminus\Delta_1$, where $\mathbf{I}\subset\mathbf{Z}^m_+(\circledast)$ is the family of multi-indices given by

$$\mathbf{I}=\{(0,\dots,0),(1,\dots,0),(1,1,\dots,0),\dots,(1,1,\dots,1),(2,1,\dots,1),\dots\}.$$

A key step in proving (27) was to obtain interlacing properties of the zeros of consecutive polynomials $Q_{\mathbf{n}}$ from which follows that the ratios $Q_{\mathbf{n+1}}/Q_{\mathbf{n}}$ form a normal family. The limit function $F$ is given in terms of certain algebraic functions of order $m+1$. However, an explicit expression of $F$ is not available for $m>1$. When $m=1$ the formula (27) reduces to the celebrated Rakhmanov theorem for orthogonal polynomials on the real line.

The polynomials $\{Q_{\mathbf{n}}\}_{\mathbf{n}\in\mathbf{I}}$, satisfy a $m+2$-term recurrence relation in the case of Angelesco and Nikishin systems. Not much is known about the behavior of its coefficients. In [A77] is shown that, in the Angelesco case, they are uniformly bounded if and only if all the intervals $\Delta_i$, $i=1,\dots,m$, are bounded. For Nikishin systems the coefficients are uniformly bounded if and only if $\Delta_1$ is bounded. Besides, they have periodic limit under the condition $\sigma_i'>0$ a. e. on $\Delta_i$, $i=1,\dots,m$. Again, the fact that the zeros of consecutive $Q_{\mathbf{n}}$ interlace plays a fundamental role.

Parallel to the progress in asymptotics, the set of strongly normal multi-indices was considerably enlarged in [A69] by including all those multi-indices for which there do not exist $1\le j<k<l\le m$ such that $n_j<n_k<n_l$. This set will be denoted by $\mathbf{Z}^m_+(*)$. In addition, the Nikishin systems made up of three functions were proved to be perfect in [A68]. A different proof of the strong normality of the multi-indices in $\mathbf{Z}^m_+(*)$ was given in [A72], where the method of the proof allowed the authors to obtain the extra interpolation points on $\Delta_2$ for the class $\mathbf{Z}^m_+(*)$. Obviously, each progress in normality implies an extension of the results concerning asymptotics. So, all three papers [A68], [A69], and [A72] contain, following [A35], applications to the convergence of the Hermite-Padé aproximants.

At that point it became natural to obtain results for the most general possible situation. So, [A75] deals with the exact rate of convergence of generalized Hermite-Padé approximants for Nikishin systems constructed with multi-indices in the set $\mathbf{Z}_+^m(*)$. It was assumed that the orthogonality relations are proportionally distributed among the components of $\mathbf{n}$ and $\sigma_1' > 0$ a. e. on $\Delta_1$. The rest of measures $\sigma_i,\ i = 2, \ldots, m$, are required to be regular in the sense of [72]. The fixed poles have even multiplicity and lie in $\Delta_1$ and the interpolation points belong to a compact subset of $\overline{\mathbf{C}} \setminus \Delta_1$. Also, the interpolation points and the fixed poles are assumed to have limit distributions. Along the same lines of [40], the $n$-th root asymptotic behavior of the polynomials $Q_{\mathbf{n},i}$ is obtained in terms of the solution of a vector valued equilibrium problem, now in the presence of an external vector field. The external field represents the influence of the established tables of interpolation points and fixed poles. Regions of convergence and divergence of the approximants $R_{\mathbf{n},i}$ are described and convergence in the largest region $\overline{\mathbf{C}} \setminus \Delta_1$ is obtained provided that the poles are adequately fixed. Roughly speaking, the fixed poles must be placed in such a way that the support of the first component of the equilibrium measure is the whole interval $\Delta_1$.

In [A82] the problem of convergence was broached from a different point of view. It was sought to prove geometric rate of convergence of multi-point Hermite-Padé approximants for Nikishin systems under as weak assumptions as possible. Again, the interpolation points belong to a compact subset of $\overline{\mathbf{C}} \setminus \Delta_1$ but the rest of the assumptions only affect the multi-indices $\mathbf{n}$. Actually, it is required that

$$n_i \geq |\mathbf{n}|/m - C|\mathbf{n}|^\kappa, \quad C > 0, \quad \kappa < 1, \quad i = 1, \ldots, m,$$

and, if $m > 3$, $\mathbf{n} \in \mathbf{Z}_+^m(*)$. So, as compared to [A75], the results are less precise in the expression of the rate of convergence but much more general in the class of measures and table of interpolation points considered.

The result of [A73] on ratio asymptotics of the polynomials $Q_{\mathbf{n}}$ was generalized in [A86] in the direction followed by Denisov [24]. He obtained an extension of Rakhmanov's Theorem to the case when the support of the measure is of the form $\Delta \cup e$, where $\Delta$ is a bounded interval, $e$ is a set without accumulation points in $\mathbf{R} \setminus \Delta$, and the Radon-Nikodyn derivative of the measure is positive a. e. on $\Delta$. The asymptotic relation (27) was proved in [A86] for measures $\sigma_i,\ i = 1, \ldots, m$, being as in Denisov's Theorem and multi-indices $\mathbf{n} \in \Lambda \subset \mathbf{Z}_+^m(*)$, where $\Lambda$ is an infinite sequence of distinct multi-indices with the property

$$\max_{\mathbf{n} \in \Lambda} \max_{i=1,\ldots,m} \{m\, n_i - |\mathbf{n}|\} < +\infty.$$

That condition means that orthogonality is, basically, equally distributed among all measures and is used to prove a weak star formula like (26) needed in the proof of the ratio convergence.

Finally, recent work is concerned with the extension of the simultaneous approximation to the setting of orthogonal expansions. The so-called Fourier-Padé approximants arose when it was necessary to represent a function given by a polynomial expansion beyond the region of convergence of the expansion. Let $\Delta_0$ be a closed interval and $l_k,\ k = 0, 1, \ldots$, the system of orthonormal polynomials with respect to a positive finite Borel measure $\sigma_0$ with support on $\Delta_0$ consisting of infinitely many points. Let $f$ be an integrable function with respect to $\sigma_0$. The linear Fourier-Padé approximant of type $(m,n)$ of the function $f$ is the rational function $R_{m,n} = p_m/q_n$ determined by

$$\deg p_m \leq m, \quad \deg q_n \leq n, \quad q_n \not\equiv 0,$$

and the condition that the first $n+m+1$ terms in the orthogonal expansion of $q_n f - p_m$ by polynomials $\{l_k\}_{k\in\mathbf{N}}$ are $0$. Several convergence results for rows of Fourier-Padé approximants when $f$ is a meromorphic function were obtained by Suetin [76] and diagonal sequences to a Markov function were studied by Gonchar, Rakhmanov, and Suetin [41]. It has been recently noted that the Fourier-Padé approximants may be used to reduce the Gibbs phenomenon [46].

A step forward is given in [A84] where simultaneous Fourier-Padé approximants for Angelesco systems are considered. The measures $\sigma_i$ are taken to be regular and the system of intervals $\Delta_i, i = 1,\ldots,m$, does not overlap $\Delta_0$. The vector of rational functions $R_{\mathbf{n}}$ is defined by

$$\deg Q_{\mathbf{n}} \leq |\mathbf{n}|, \quad Q_{\mathbf{n}} \not\equiv 0, \quad \deg P_{\mathbf{n},i} \leq |\mathbf{n}| - 1,$$

and the requirement that the first $\mathbf{n} + n_i$ terms in the orthogonal expansion of $Q_{\mathbf{n}}\,\widehat{\sigma}_i - P_{\mathbf{n},i}$ by polynomials $\{l_k\}$ are $0$, $i = 1,\ldots,m$.

In [A84] it is proved that $R_{\mathbf{n}}$ is uniquely determined by the above conditions with $Q_{\mathbf{n}}$ having $n_i$ simple zeros in $\Delta_i$, $i = 1,\ldots,m$. Again, the key fact is the appearance, for each $R_{\mathbf{n},i}$, of $|\mathbf{n}| + n_i$ interpolation points on the interval $\Delta_0$ resulting in $Q_{\mathbf{n}}$ verifying varying multi-orthogonality relations. Following the scheme of [40], the exact rate of convergence and divergence of the approximants for arbitrary $\mathbf{n} \in \mathbf{Z}_+^m$ is given in terms of the solution to an associated vector valued equilibrium problem. Results for non-linear simultaneous Fourier-Padé approximants are also given. They are constructed by requiring the first $\mathbf{n} + n_i$ terms in the orthogonal expansion of $\widehat{\sigma}_i - P_{\mathbf{n},i}/Q_{\mathbf{n}}$ to be $0$. The results are analogous, except for the fact that the uniqueness of the non-linear approximants is not guaranteed. Current work is concerned with Fourier-Padé approximants for Nikishin systems.

In the next section some applications of the Hermite-Padé approximants to simultaneous quadrature formulas will be presented.

# 7 Quadrature rules.

It may be said that the use of polynomials orthogonal with respect to varying measures is a common feature of most of Guillermo's work and his contributions to the study of quadrature formulas were not to be an exception to that principle. Rational quadrature rules, i.e., quadrature rules that exactly integrate rational functions, were introduced by Guillermo in [A11] and [A16] but they already implicitly appear in his first works [A1, A2, A3] connected to multipoint Padé approximation and thus to orthogonal polynomials with respect to varying measures. Other contributions deal with Gauss-Kronrod formulas, interpolatory rules with complex weight functions, simultaneous quadrature rules, quadrature formulas for unbounded intervals, etc... Most of them are related to formulas of the highest degree of exactness, that is, Gaussian-like rules. For an account of the computational aspects of Gaussian-type rules, see [33]. Additional information on all kinds of quadrature formulas may be found in [23, 51].

Let $\mu$ be a finite positive Borel measure whose support $\Sigma$ consists of an infinite set of points contained in $\mathbf{R}$. If $\Sigma$ is not a bounded set, we will assume that the moments of the measure $\mu$ are all finite numbers. The smallest interval containing $\Sigma$ will be denoted by $\Delta$. Let us consider the integral

$$I(f) = \int f(x)\, d\mu(x).$$

A quadrature rule for the integral above is any linear functional of the form

$$I_n(f) = \sum_{i=1}^{n} \lambda_{n,i}\, f(x_{n,i}), \quad \lambda_{n,i} \in \mathbf{C}, \quad i = 1, \dots, n.$$

The nodes $\{x_{n,i}\}$ are usually required to belong to $\Delta$. The rule $I_n$ is called positive if all the numbers $\lambda_{n,i}$ are positive. The error of the rule, $I(f) - I_n(f)$, will be denoted by $E_n(f)$.

Typically, the nodes $\{x_{n,i}\}$ and the weights $\{\lambda_{n,i}\}$ are chosen so that $I_n(f)$ is a good approximation of $I(f)$ and/or convergence of $I_n(f)$ to $I(f)$ as $n$ tends to infinity is obtained for large classes of integrands. Also, for convergence and numerical stability reasons, positivity of the quadrature weights $\{\lambda_{n,i}\}$ is an important and often sought property. Even so, there is room for manoeuver and a central problem in the theory is how to choose nodes and weights in the most convenient way. Two different approaches may be pointed up. In one of them, due to Kolmogorov, a quadrature rule is constructed having minimal error on a given class of functions. That is, let $A$ be a class of functions, usually with low order of regularity. Then, the best quadrature rule, in this sense, is that for which $\sup\{|E_n(f)| : f \in A\}$ attains its minimum; see [13] for details and references.

More frequently, a quadrature rule is built so as to be exact in a set of basis functions which satisfy good properties like being very simple and spanning, as $n$ goes to infinity, a set which is dense in a space of continuous functions. The prototype is $\mathbf{P}_n$, the set of polynomials of degree less than or equal to $n$. If the nodes $\{x_{n,i}\}$ are given, the weights $\{\lambda_{n,i}\}$ may be chosen so that the rule $I_n$ is exact, at least, in $\mathbf{P}_{n-1}$; such a rule is called interpolatory because $I_n(f) = I(L_n(f))$, where $L_n(f)$ is the interpolating polynomial of $f$ at the nodes $\{x_{n,i}\}$.

Let $\hat{\mu}(z)$ be the Cauchy transform of the measure $\mu$ with $\Delta = [0, +\infty)$. Let us consider the expansion of $\hat{\mu}(z)$ in a Chebyshev continued fraction

$$\cfrac{a_1^2}{z - b_1 - \cfrac{a_2^2}{z - b_2 - \dots}},$$

whose $n$th convergent $P_n/Q_n$ holds

$$\hat{\mu}(z) - \frac{P_n(z)}{Q_n(z)} = o\left(\frac{1}{z^{2n}}\right), \quad z \to \infty. \tag{28}$$

It is well known that the polynomials $P_n$ and $Q_n$ satisfy the same second order difference equation so the convergent $P_n/Q_n$ admits the decomposition

$$\frac{P_n(z)}{Q_n(z)} = \sum_{i=1}^{n} \frac{A_{n,i}}{z - z_{n,i}}, \quad A_{n,i} > 0, \quad i = 1, \ldots, n. \tag{29}$$

It was Gauss who first realized that the maximum degree of exactness for interpolatory rules is attained when the nodes are taken to be the zeros of $Q_n$. Due to (28) and (29), the quadrature weights turn out to be the residues $\{A_{n,i}\}$, known as the Christoffel numbers. The corresponding quadrature rule $I_n^G$, called Gaussian, is exact in $\mathbf{P}_{2n-1}$. Therefore, Gaussian rules may be considered as optimal among interpolatory rules.

As a result of (28), the convergent $P_n/Q_n$ is precisely the $n$th diagonal Padé approximant of the function $\hat{\mu}(z)$, establishing a key relation between rational approximation and quadrature rules. On one hand, expression (29) implies that the sequence of Padé approximants is normal, from which the Stieltjes Theorem follows under suitable conditions. On the other hand, by the same token, we have

$$\hat{\mu}(z) - \frac{P_n(z)}{Q_n(z)} = E_n\left(\frac{1}{z - (\cdot)}\right).$$

The Stieltjes Theorem therefore implies the convergence of the Gaussian rules to the integral for all functions $1/(z - (\cdot))$, $z \notin [0, +\infty)$. Additionally, the linear functionals $I_n^G$ are uniformly bounded since the quadrature weights are all positive. Now, a density argument proves convergence of the rules for all functions that are continuous on $[0, +\infty)$ with finite limit at $+\infty$. Thus, convergence of the sequence of Padé approximants to Cauchy transforms is equivalent to convergence of the Gaussian rules for a large class of continuous functions. This is a particular case of one of the main results of [A11].

Furthermore, suppose that $\Delta$ is a bounded interval and $f$ an analytic function on a neighborhood of $\Delta$. A standard application of Cauchy's Integral Formula and Fubini's Theorem gives the representation

$$E_n(f) = \frac{1}{2\pi i} \int_\gamma f(\zeta) \left(\hat{\mu}(\zeta) - \frac{P_n(\zeta)}{Q_n(\zeta)}\right) d\zeta, \tag{30}$$

where $\gamma$ is any positively oriented simple curve surrounding the interval $\Delta$ and contained in the region of analyticity of $f$. It is clear that

$$\limsup_{n\to\infty} |E_n(f)|^{1/n} \le \limsup_{n\to\infty} \|\hat{\mu} - P_n/Q_n\|_\gamma^{1/n}.$$

Formula (30) therefore allows us to obtain estimates of the speed of convergence of Gaussian quadrature rules for analytic functions from results about the rate of convergence of the Padé approximants to Markov functions.

Before referring to those papers by Guillermo strictly dedicated to quadratures, let us say that rational approximation to Cauchy transforms and Gaussian-like quadrature rules turn out to be inextricable subjects in his work. Most results of the former class have application to the latter although it may not appear in the paper or it is developed later on. To set an example, the results on the speed of convergence of two-point Padé approximants to Stieltjes functions of [A43] are applied in [15] to obtain estimates of the rate of convergence of Gaussian-type rules which are exact in a space of Laurent polynomials, provided that the integrand is analytic on a neighborhood of $[0, +\infty)$. Conversely, certain techniques based on quadrature rules are often needed in the course of a proof even though they are not explicitly referred to in the statement of the main theorem. As was mentioned, that happens from the first works [A1, A2, A3] on. In [A10] a quadrature rule of nearly optimal degree of exactness is used to estimate the size of a modified approximation error which is a key step in proving convergence of the Padé approximants to a meromorphic Stieltjes function. Other examples are [A65], where convergence of multipoint Padé-type approximants to a meromorphic Markov function is proved by using a similar reasoning, or [A82] on the convergence of multipoint Hermite-Padé approximants to Nikishin systems.

The existence of formulas based on $n$ nodes and exact in a $2n$-dimensional vector space is not limited to the case mentioned of $\mathbf{P}_{2n-1}$. Krein [49] proved that given any Chebyshev system of $2n$ continuous functions on $[-1, 1]$, there exists a unique couple of sets of $n$ nodes contained in $(-1, 1)$ and $n$ strictly positive quadrature weights such that the corresponding quadrature rule is exact in the system and that gives the maximum degree of exactness. Other generalizations allowing multiple nodes are [48, 10]. One of the main examples of this more general situation is when the space where the quadrature rule is exact is formed by rational functions.

The first paper by Guillermo dealing exclusively with rational quadrature rules was published in 1982. Ten years later rational formulas attracted the attention of Gautschi [29, 30] and van Assche [80]. Numerical experiments showed that they give good results if the poles of the rational functions used to build the rule are chosen so as to simulate the singularities of the function to be integrated. Another important property is that, according to the results of [39], the nodes of the rational Gaussian rules tend to the equilibrium measure in the presence of the external field generated by the limit distribution of the poles of the rational functions. These facts have stimulated the interest in rational rules until today: computer routines implementing rational Gaussian rules are designed in [31], extensions to Gauss-type rules may be found in [32], and in [17] an error bound for analytic integrands is given. Much study has been dedicated to the particular case of quadrature formulas that are exact in spaces of Laurent polynomials. They mostly appear in connection with integration either on unbounded real intervals [16] or on the unit circle [22].

Let us describe the rational Gaussian rules. Let $\mathbf{A} = \{\alpha_{n,i},\, i = 1, \ldots, 2n,\, n \in \mathbf{N}\} \subset \overline{\mathbf{C}} \setminus \Delta$ be a table of points such that the polynomials

$$w_{2n}(x) = \prod_{i=1}^{2n} \left(1 - \frac{x}{\alpha_{n,i}}\right), \quad \deg w_{2n} \leq 2n - 1, \quad n \in \mathbf{N}, \tag{31}$$

have real coefficients. We will say that the table $\mathbf{A}$ is Newtonian if, for all $n \in \mathbf{N}$, $\alpha_{n,i} = \alpha_i$. By $Q_n^{\mathbf{A}}$ we denote the $n$th orthogonal polynomial with respect to the varying measure $d\mu/w_{2n}$. Let $x_{n,i}$, $i = 1, \ldots, n$, be the zeros of $Q_n^{\mathbf{A}}$ and $A_{n,i}$ the Christoffel numbers corresponding to the Gauss quadrature formula for the measure $d\mu/w_{2n}$. Then, the quadrature rule given by

$$I_n^{\mathbf{A}}(f) = \sum_{i=1}^{n} \lambda_{n,i}\, f(x_{n,i}), \quad \lambda_{n,i} = w_{2n}(x_{n,i})\, A_{n,i}, \quad i = 1, \ldots, n,$$

satisfies $I(p/w_{2n}) = I_n^{\mathbf{A}}(p/w_{2n})$ for any polynomial $p \in \mathbf{P}_{2n-1}$. Thus, the rule $I_n^{\mathbf{A}}$ is exact in a $2n$-dimensional vector space of rational functions and is called a rational Gaussian rule. Rational Gaussian rules share with the classical Gaussian one the most basic and important properties, namely, they are of the highest degree of exactness, their nodes are simple and lie in $\Delta$, and the weights $\lambda_{n,i}$ are positive numbers.

As in the polynomial case, if the nodes of the rule are specified in a different way, it is still possible to choose the quadrature weights so that the rule is exact, at least, in a $n$-dimensional vector space of rational functions and then it is known as a rational rule of interpolatory type.

One of the classical results in this field is the Steklov Theorem on the convergence of quadrature formulas for Riemann integrable functions. An extension to rational Gaussian rules for unbounded intervals of the Steklov Theorem was given in [A11]. The table $\mathbf{A}$ was taken to be Newtonian and $\Delta = [0, +\infty)$. The result was obtained under a condition which assures the family

$$\left\{1, \frac{1}{1 - x/\alpha_1}, \ldots, \frac{1}{1 - x/\alpha_n}, \ldots\right\}$$

to span a dense set in the space of functions that are continuous on $[0, +\infty)$ with finite limit at $+\infty$. It was also proved that the convergence of the rational Gaussian rules for that class of continuous functions is equivalent to the convergence of the multipoint Padé approximants to the Stieltjes function $\widehat{\mu}$. As a consequence, the conditions imposed in [A3] on Newtonian tables of points for convergence of such approximants were weakened.

The above-mentioned result about equivalence between different kinds of convergence was generalized in [A23] by considering any positive rational rule of interpolatory type exactly integrating constants.

The rest of Guillermo's papers on quadrature formulas deal with the interval $\Delta$ being bounded. Rather than following a chronological order, let us group them by subject.

An estimate of the rate of convergence of rational Gaussian rules for analytic integrands was obtained in [A16]. Specifically, let $f$ be an analytic function on a neighborhood $\Omega$ of $\Delta$ and suppose that $\mathbf{A}$ is contained in a compact set $K$ of $\overline{\mathbf{C}} \setminus \Omega$, then

$$\limsup_{n\to\infty} |E_n^{\mathbf{A}}(f)|^{1/2n} \le e^{-\tau}, \qquad \tau = \inf\left\{g_{\overline{\mathbf{C}}\setminus\Sigma}(z,\zeta),\ z \in \partial\Omega,\ \zeta \in K\right\}, \tag{32}$$

where $g_{\overline{\mathbf{C}}\setminus\Sigma}$ stands for the generalized Green function with respect to the region $\overline{\mathbf{C}} \setminus \Sigma$. The main ingredients of the proof are the analog representation of (30) for multipoint Padé approximants and the results of [A2] on the rate of convergence of such approximants to a Markov function. The rate of convergence may be improved if the table $\mathbf{A}$ is taken to be extremal in a certain sense with respect to the sets $\Sigma$ and $\overline{\mathbf{C}} \setminus \Omega$.

As the results on Padé approximation to Markov functions were being improved and generalized, the corresponding results on quadratures followed the same trail. Both works [A50] and [A62] are concerned with the same very general scheme of rational rule. Let us describe the latter, which represents a more evolved solution with some ideas borrowed from [1]. The table $\mathbf{A}$ is taken to be contained in a compact set of $\overline{\mathbf{C}}\setminus\Delta$. Some nodes of the quadrature rule are preassigned at the zeros of the polynomial $L_n^2$, $\deg L_n = k(n) \leq n$, which belong to $\Delta$. Denote by $\mathsf{B}$ the corresponding table of fixed nodes. For each $n \in \mathbf{N}$, the total amount of nodes will be $n + k(n)$. To achieve the maximum degree of exactness which turns out to be $2n$ in all cases, the rest of the nodes must be the zeros of the polynomial $Q_{n-k(n)}^{\mathsf{A},\mathsf{B}}$ of degree $n - k(n)$ orthogonal with respect to the varying measure $L_n^2 d\mu/w_{2n}$. This way of proceeding allows the results to be displayed independently of the ratio $k(n)/n$. Obviously, unless $k(n) = 0$, the quadrature rule constructed in this fashion will have multiple nodes and will be of the form

$$I_n^{\mathsf{A},\mathsf{B}}(f) = \sum_{i=1}^{N} \sum_{j=0}^{M_i} \lambda_{n,i,j}\, f^{(j)}(x_{n,i}).$$

Let $\alpha$ be the weak star limit of the normalized zero counting measures of the polynomials $w_{2n}$ and $\lambda$ the equilibrium measure in the presence of the external field given by (minus) the logarithmic potential of $\alpha$. Three essential conditions are imposed. The support $\Sigma$ is a regular set with respect to the Dirichlet problem, the measure $\mu$ is regular in the sense of Stahl and Totik [72], and

$$\limsup_{n\to\infty} \frac{1}{n} B_n \leq \lambda,$$

where $B_n$ is the zero counting measure of the polynomial $L_n$. That condition means that the limit distribution of the fixed poles must be smaller, in the weak star sense, than $\lambda$ in order to allow the free nodes to fill the gap between the two measures. Actually, under these conditions, the normalized zero counting measures of the polynomials $L_n\, Q_{n-k(n)}^{\mathsf{A},\mathsf{B}}$ tend to the equilibrium measure $\lambda$, whence

$$\limsup_{n\to\infty} |E_n^{\mathbf{A},\mathbf{B}}(f)|^{1/2n} \leq e^{-\tau}, \qquad \tau = \inf\left\{ \int g_{\overline{\mathbf{C}}\setminus\Sigma}(z,\zeta)\, d\alpha(\zeta),\ z \in \partial\Omega \right\}, \tag{33}$$

provided $f$ is an analytic function on a neighborhood $\Omega$ of $\Delta$. This asymptotic estimate should be compared to (32).

Convergence of polynomial quadrature formulas for complex weight functions was proved in [A44]. Here the measure $\mu$ is taken to be complex with finite total variation and $\Delta = [-1, 1]$. The main difficulty in order to construct a Gaussian rule is that, in general, the location of the zeros of the polynomials orthogonal with respect to complex measures is not known. Because of it, the class of measures is restricted to those being in a complex extension of the so-called Nevai-Blumenthal class $M(0,1)$. That is, the corresponding orthogonal polynomials $Q_n$ satisfy the following three-term recurrence relation with complex coefficients

$$Q_{n+1} = (z - b_n)\, Q_n(z) - a_n^2\, Q_{n-1}(z), \quad Q_0(z) = 1, \quad Q_1(z) = z - b_0,$$

with $\lim_{n\to\infty} a_n = 1/2$ and $\lim_{n\to\infty} b_n = 0$. It is known that polynomials $Q_n$ in this class satisfy ratio asymptotics (see [A40]) whence it follows that their zeros only may accumulate on $[-1,1]$. Then, for sufficiently large $n$, it is possible to define a Gaussian quadrature rule $I_n^G$ for analytic integrands with possibly multiple nodes and prove the estimate

$$\limsup_{n\to\infty} |E_n^G(f)|^{1/2n} \le e^{-\tau}, \qquad \tau = \inf\left\{ g_{\overline{\mathbf{C}}\setminus\Delta}(z,\infty),\ z\in\partial\Omega \right\}, \tag{34}$$

where $\Omega$ is the region of analyticity of $f$. If the measure $\mu$ does not belong to the complex Nevai-Blumenthal class it is still possible to choose the nodes as the zeros of any polynomial satisfying regular $n$th root asymptotic behavior on $[-1,1]$ (cf. [72]) to define an interpolatory quadrature rule for which an estimate like (34) is obtained with speed of convergence geometric of order $n$ instead of $2n$.

The work [A44] also contains convergence results on interpolatory rules for continuous functions whenever the nodes are chosen in the following manner. Suppose that $d\mu(x) = \omega(x)dx$ and let $h$ be any weight function on $[-1,1]$ such that

$$\int_{-1}^{1} \frac{|\omega(x)|^2}{h(x)}\, dx < +\infty. \tag{35}$$

Condition (35) was introduced in connection with results on product integration rules by Sloan and Smith [70]. Now, the nodes of the rule are taken to be the zeros of the $n$th polynomial orthogonal with respect to the measure $h(x)dx$. The size of $\sum_{i=1}^{n} |A_{n,i}|$, where $\{A_{n,i}\}$ are the quadrature weights, is estimated by using (35) in terms of the analogous amount of the Gaussian rule corresponding to $h(x)\,dx$. It is then proven that $\sum_{i=1}^{n} |A_{n,i}| \le M\sqrt{n}$, where $M$ is an absolute constant. As a consequence, convergence of the rule is obtained for continuous functions with modulus of continuity of order $O(\delta^p)$, $p > 1/2$. Finally, several numerical examples are provided.

Similar results are proved in [A47] for rational rules. We only mention what is different from [A44]. Gaussian-type rules are not considered because results on ratio asymptotics for polynomials satisfying a varying three-term recurrence relation with complex coefficients are not available. Regarding interpolatory rules for analytic integrands, the nodes are fixed so that their limit distribution is the equilibrium measure in the presence of the external field given by the limit distribution of the poles of the rational functions. Such poles are compactly contained in the set $\overline{\mathbf{C}} \setminus [-1,1]$. Under this condition, an estimate of type (33) is obtained with a geometric rate of convergence of order $n$. As for continuous integrands, condition (35) is used again and the nodes are chosen as the zeros of the $n$th polynomial orthogonal with respect to the varying measure $h(x)dx/w_n$, where $w_n$ has the same meaning as in (31) with $\deg w_n \le n$. A result analogous to that of [A44] is obtained. To that end, a Jackson-type theorem is provided which estimates the error of best approximation of a continuous function $f$ by the rational functions used in the quadrature. The estimate is expressed in terms of the modulus of continuity of $f$.

The works [A66] and [A71] deal with the so-called Gauss-Kronrod quadrature rule, [A71] being the rational counterpart of [A66]. Such a rule arose from the need to estimate simultaneously an approximate value of the integral and the error as well (cf. [50]). It takes the form

$$I_{2n+1}^{GK}(f) = \sum_{i=1}^{n} A_{n,i}\, f(x_{n,i}) + \sum_{j=1}^{n+1} B_{n,j}\, f(y_{n,j}), \quad I(f) = I_{2n+1}^{GK}(f) + E_{2n+1}^{GK}(f),$$

where $\{x_{n,i}\}_{i=1}^n$ are the Gaussian nodes while the rest of the nodes $\{y_{n,j}\}_{j=1}^{n+1}$ and the quadrature weights $\{A_{n,i}\}_{i=1}^n, \{B_{n,j}\}_{j=1}^{n+1}$, are chosen so that the rule has the highest possible degree of polynomial exactness, which is $3n+1$. This requirement is equivalent to the fact that the nodal polynomial $S_{n+1}(x) = \prod_{j=1}^{n+1}(x - y_{n,j})$ satisfies the orthogonality relations

$$\int x^k\, S_{n+1}(x)\, Q_n(x)\, d\mu(x) = 0, \quad k = 0, 1, \ldots, n,$$

where $Q_n$ is the $n$th monic orthogonal polynomial with respect to $\mu$. The polynomials $S_{n+1}$ were introduced by Stieltjes [43] in 1894 for the case of the Legendre measure $d\mu(x) = dx$ and are named after him. Despite the fact that they are orthogonal with respect to a non-positive measure, their zeros occasionally behave, depending on $\mu$, as those of the classical orthogonal polynomials, which implies the existence of the corresponding Gauss-Kronrod rule. The most studied case has been that of the ultraspherical measures for which the Stieltjes polynomials show different kinds of behavior, see [77, 64, 18]. When they exist, Gauss-Kronrod rules can be computed efficiently [52] and are used in packages for automatic integration [65]. For further details on Stieltjes polynomials and Gauss-Kronrod quadrature formulas, see the surveys [28, 58].

In [A66], the asymptotic properties of the Stieltjes polynomials outside the support of the measure $\mu$ are studied. For that, an integral expression connecting the Sieltjes polynomials and the functions of the second type is derived. The main result is that, when the support of $\mu$ is an interval $\Delta$, the asymptotic behavior of the polynomials $\{S_{n+1}\}_{n\in\mathbf{N}}$ is similar to that of standard orthogonal polynomials. Thus, weak, ratio, and strong asymptotics is obtained for $\mu$ being regular, in the Nevai class, and in the Szegő class, respectively. As a consequence, for regular measures, the zeros of the Stieltjes polynomials, although they may be complex, tend to the interval $\Delta$ and their limit distribution is the equilibrium measure of $\Delta$. This implies that a Gauss-Kronrod quadrature rule with possibly multiple nodes always exists for analytic integrands. For such generalized rules an estimate of type (34) is given with $|E_{2n+1}^{GK}(f)|^{1/3n}$ asymptotically bounded by $e^{-\tau}$.

If the support of $\mu$ is not an interval a distinct phenomenon can be observed. The support $\Sigma$ is a proper subset of its convex hull $\Delta$ and the existence of zeros of the functions of the second type in $\Delta \setminus \Sigma$ may cause regions of divergence of the polynomials $S_{n+1}$ to appear in a neighborhood of $\Sigma$.

The main difficulty in extending the above results to rules exact in a vector space of rational functions of dimension $3n+2$ was to define the corresponding Stieltjes polynomials in a proper way. The following definition was given in [A71]. For each $n \in \mathbf{N}$, let $w_{2n}$ be a polynomial as in (31) but with $\deg w_{2n} \leq 2n+1$. Let $Q_{n,k}$ be the $k$th monic orthogonal polynomial with respect to the varying measure $d\mu/w_{2n}$. In addition, let $\{v_n\}$ be another family of polynomials with the same features of $\{w_{2n}\}$ and $\deg v_n \leq n+1$, $n \in \mathbf{N}$. Then, the Stieltjes polynomial $S_{n,k+1}$ is the monic polynomial of least degree verifying

$$\int x^\nu \, S_{n,k+1}(x) \, Q_{n,k}(x) \, \frac{d\mu(x)}{w_{2n}(x) \, v_n(x)} = 0, \quad \nu = 0, 1, \ldots, k.$$

It may be proved that $\deg S_{n,k+1} = k+1$ if $k+1 \geq \deg v_n$. For these varying Stieltjes polynomials analogous results to those of [A66] were obtained in [A71] with the limit distribution of the poles of the rational functions playing the usual role. In particular, an estimate of the form (33) is given. Current work is concerned with the extension of the Gauss-Kronrod scheme to quadrature rules on the unit circle.

In the rest of the section, we will adopt the notation used in the section about Hermite-Padé approximants. Simultaneous quadrature formulas deal with the problem of evaluating several integrals with a common integrand

$$I^i(f) = \int f(x) \, ds_i(x), \quad i = 1, \ldots, m,$$

by means of a quadrature rule with the same set of nodes. That is,

$$I^i(f) = I^i_n(f) + E^i_n(f), \quad I^i_n(f) = \sum_{j=1}^{n} A_{n,i,j} f(x_{n,j}), \quad i = 1, \ldots, m.$$

Simultaneous quadrature rules were introduced by Borges [12] although they implicitly appear in the work by Nikishin [61]. They are the object of study in the papers [A72] and [A81].

We will say that the rule $\mathsf{I}_n = (I^1_n \ldots, I^m_n)$ is interpolatory if the rules, $I^i_n$, $i = 1, \ldots, m$, are exact at least in the space $\mathsf{P}_{n-1}$. Suppose that the multi-index $\mathbf{n}$ is strongly normal. Choose the nodes as the zeros of the common denominator $Q_{\mathbf{n}}$ of the Hermite-Padé approximant $R_{\mathbf{n}}$ to the system of Markov functions determined by $S = (s_1, \ldots, s_m)$. Then, the simultaneous quadrature rules $I^i_{|\mathbf{n}|}$ are exact in $\mathsf{P}_{|\mathbf{n}|+n_i-1}$, $i = 1, \ldots, m$, and $\mathsf{I}_{|\mathbf{n}|}$ is interpolatory. With this choice of nodes, convergence of the simultaneous rule $\mathsf{I}_{|\mathbf{n}|}$ for integrable, continuous, Lipschitz, and analytic functions is obtained in [A72] under hypothesis depending on the size and properties of the quadrature weights $\{A_{|\mathbf{n}|,i,j}\}$. In particular, for multi-indices of the form $\mathbf{n} = (0, \ldots, n, \ldots, 0)$, where the number $n$ is placed always in the same $k$th component of $\mathbf{n}$, it is proved that all the quadrature weights $A_{|\mathbf{n}|,k,j}$ have the same sign, therefore obtaining convergence of $I^k_{|\mathbf{n}|}$ for Riemann integrable functions. Additionally, if another measure $s_{k'}$ satisfies a subordination condition of the type (35) with respect to $s_k$, convergence of $I^{k'}_{|\mathbf{n}|}$ for continuous functions is obtained.

The second part of [A72] is dedicated to the special case of the system of measures $S$ being a Nikishin system. Here, partial results on the sign-preserving property of the quadrature weights corresponding to some components of the rule $\mathsf{I}_{|\mathbf{n}|}$ are given. If the orthogonality conditions are nearly equally distributed between all the measures, i.e., $n_i \geq |\mathbf{n}|/m - C$, $i = 1, \ldots, m$, and the sequence of multi-indices $\Lambda$ is contained in $\mathbf{Z}^m_+(*)$, geometric rate of convergence of $\mathsf{I}_{|\mathbf{n}|}$ for functions analytic on a neighborhood of $\Delta_1$ is obtained.

The extension of the above results to simultaneous rational quadrature rules is carried out in [A81]. The construction is limited to measures $ds_i(x) = w_i(x)\,ds(x)$, where $w_i$, $i = 1, \ldots, m$, are weight functions and $s$ is a finite positive Borel measure on a bounded interval $\Delta$. Having fixed a table **A** of interpolation points, the nodes are now taken as the zeros of the common denominator of the Hermite-Padé approximant which interpolates the system of functions $\widehat{S}$ at the points of **A**. Similar results to those of [A72] are given with, occasionally, some conditions imposed on **A** like being compactly contained in $\overline{\mathbf{C}} \setminus \Delta$ or giving rise to rational functions holding a density property in the space of functions continuous on $\Delta$. Emphasis is put in this paper on the numerical side, providing a procedure to compute the simultaneous rules for systems made up of three functions and for a particular choice of multi-indices. The method combines the rational nature of the rules with smoothing changes of variable to ease the intermediate calculations and is suitable when the integrand has real poles near $\Delta$. Some numerical experiments show the power of the method.

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